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Graceful Labeling of Some Spider Graphs

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Abstract. A graceful labeling of a tree T with n edges is a bijection $f : V(T) \longrightarrow \{0, 1, 2, \ldots n\}$ such that $\{|f(u) - f(v)| : uv \in E(T)\}$ equal to $\{1, 2, 3, \ldots, n\}$. A spider graph is a tree with one vertex of degree at least 3 and all others with degree at most 2. We show that some classes of spider graphs admit graceful labeling.

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Key Words and Phrases: Spider graph, graceful labeling, inverse transformation

1. Introduction

Labeled graphs form useful models for a wide range of disciplines and applications such as in coding theory, X-ray crystallography, radar, astronomy, electronic circuit design and communication network addressing [1]. A systematic presentation of diverse applications of graph labeling is presented in [2].

A graceful labeling of a tree is a special kind of labeling graph, i.e., a graceful labeling f of a tree T is a bijective function from the set of vertices $V(T)$ of T to the set $\{0, 1, 2, \ldots, |E(T)|\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. There are other concepts of labeling graphs that are equivalent to graceful labeling, for instance, edge antimagic vertex labeling and rainbow antimagic labeling (see [3], [4]). A tree that admits graceful labeling is called a graceful tree. In 1964, Rangel and Rosa (see [5], [6]) gave the famous and unsolved **graceful tree conjecture** which stated that **all trees are graceful**. In order to solve the graceful tree conjecture, we also start to consider a graceful labeling of some graph which usually belongs to a subgraph of a tree such as we called a spider graph. The results of graceful labeling on spider graphs is a step towards potentially solving the graceful tree conjecture.

A spider graph is a tree with one vertex of degree at least 3 and all others with degree at most 2. Gillian [1] has noted that the special case of the conjecture regarding a spider

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graph is still open and that very few classes of spider graphs are known to be graceful. Huang et al. [7] proved that all spider graphs with three or four legs are graceful. Bahls et al. [8] also proved that every spider graph in which the lengths of any two of its legs differ by at most one is graceful. Jampachon et al. [9], [10] have also proven that $S_n(k, l, m)$ is graceful, where $S_n(k, l, m)$ is defined in Section 2. Recently, Panpa et al. [11] proved that all spider graphs with at most four legs of lengths greater than one are graceful.

To prove our results, we need some terminology and existence results which are described below.

Let T be a tree with n edges. A *graceful labeling* of T is a bijection $f : V(T) \to \{0, 1, 2, \dots, n\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the edge label set is equal to $\{1, 2, 3, \dots, n\}$. A tree with a graceful labeling is called a *graceful tree*. In [12] Hrnčiar and Haviar proved Lemma 1 and in [10] Jampachon and Poomsa-ard proved Lemma 2 and Lemma 3.

Lemma 1. *Let T be a tree with n edges and a graceful labeling f. Then, the function $f^* : V(T) \to \{0, 1, 2, \dots, n\}$ is given $f^*(v) = n - f(v)$ is also a graceful labeling of T.*

Lemma 2. *Let P_{2n} be a path graph with $V(P_{2n}) = \{v_1, v_2, v_3, \dots, v_{2n}\}$ and let $M = \{a+1, a+2, \dots, a+n, m+a+1, m+a+2, \dots, m+a+n\}$, $m \geq n$ and $a \geq 0$. Then, there is a bijective labeling $f : V(P_{2n}) \to M$ such that $f(v_1) = i$ or $f(v_{2n}) = i$, where $i \in M$ and the edge label set is $\{m-n+1, m-n+2, \dots, m+n-1\}$.*

Lemma 3. *Let P_n be a path graph with $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and let $M = \{m, m+1, m+2, \dots, m+n-1\}$. Then, there is a bijective labeling $f : V(P_n) \to M$ such that $f(v_1) = i$ or $f(v_n) = i$, where $i \in M$ and the edge label set is $\{1, 2, 3, \dots, n-1\}$.*

Let T be a tree and let u be a leaf of T. Let $T(u, u_1, u_2, \dots, u_n)$ be the tree obtained from T by adding the vertices $u_1, u_2, u_3, \dots, u_n$ and the edges $uu_1, u_1u_2, \dots, u_{n-1}u_n$. In [13] Sangsura and Poomsa-ard have proved Lemma 4.

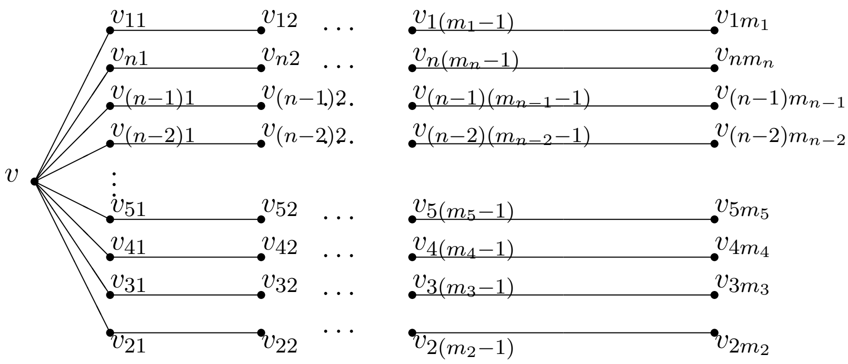
Lemma 4. *If a tree T has a graceful labeling f such that $f(u) = 0$, where u is a leaf of T, then $T(u, u_1, u_2, \dots, u_n)$ is graceful for $n \geq 1$.*

A *spider graph* or *spider* is a tree with at most one vertex of degree greater than 2 and this vertex is called the *branch vertex* and is denoted by v_0. A *leg* of a spider graph is a path from the branch vertex to a leaf of the tree. Let $S_n(m_1, m_2, \dots, m_k)$, $n \geq k$, denote a spider graph of n legs such that its legs have length one except for k legs of lengths $m_1, m_2, \dots, m_k$, where $m_i \geq 2$ for all $i = 1, 2, \dots, k$. In [11] Panpa and Poomsa-ard proved Lemma 5 and Lemma 6.

Lemma 5. *If $S_k(m_1, m_2, \dots, m_k)$ has a graceful labeling f such that $f(v_0) = 0$, then there is a graceful labeling f' of $S_n(m_1, m_2, \dots, m_k)$ such that $f'(v_0) = 0$.*

Lemma 6. *If $S_k(m_1, m_2, \dots, m_k)$ has a graceful labeling f such that $f(v_0) = 0$, then $S_n(l, m_1, m_2, \dots, m_k)$ is also graceful.*

In this paper we study graceful labeling of some class of spider graphs whose graceful labeling is generated by our two special labeling types. Now, we consider the special labeling of a graph as the following. Let T be a spider graph of n legs and for each i-th leg of T is represented by a path $v - v_{im_i}$ where an integer $m_i \geq 2$ for all $i = 1, 2, \ldots, n$ as shown in Figure 1. It can be seen that $|V(T)| = m_1 + m_2 + m_3 + \ldots + m_n + 1$. We now introduce a labeling f of T, which will be used to generate graceful labelings in the following subsequent part.



. Figure 1

Let T' be obtained from T by deleting the edges $vv_{31}, vv_{41}, \ldots, vv_{(n-1)1}, vv_{n1}$ and adding the edges $v_{2m_2}v_{31}, v_{3m_3}v_{41}, \ldots, v_{(n-2)m_{(n-2)}}v_{(n-1)1}, v_{(n-1)m_{(n-1)}}v_{n1}$. Then we get T' is a path. Next we will label the vertices of path T' in the following way.

Case m_1 is odd. Let $m_1 = 2p + 1$. Then we label the vertices of T' alternating between the highest and the lowest remaining unused labels by starting with $f(v_{1m_1}) = m_1 + m_2 + \ldots + m_n$ and $f(v_{1(m_1-1)}) = o$. Then we get $f(v) = p$.

Case m_1 is even. Let $m_1 = 2p'$. Then we label the vertices of T' alternating between the lowest and the highest remaining unused labels by starting with $f(v_{1m_1}) = 0$ and $f(v_{1(m_1-1)}) = m_1 + m_2 + \ldots + m_n$. Then we get $f(v) = p'$.

Notice that the labeling in two cases of T' are graceful. We will use the labeling in two cases to generate graceful labeling of some spider graphs in the later part. For convenience, we call the labeling constructed in the first case and the second case as Type I and Type II, respectively.

2. Main Results

In this section we want to show that some spider graphs are graceful. At first, for a spider graph that has n legs and there exist $n - 2$ legs of them are even lengths as the following theorem.

Theorem 1. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If $m_{i+1} = 2m_i$ for $i = 2, 3, \ldots, n - 1$, then T is a graceful labeling graph.*

Proof. Let T be the spider as shown in Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. Consider $|f(v_{31}) - f(v_{2m_2})|$. Since $m_3 = 2m_2$, then by the way of labeling of T of Type I there exists $v' \in \{v_{31}, v_{32}, \ldots, v_{3m_3}\}$ such that $|f(v') - f(v)| = |f(v_{31}) - f(v_{2m_2})|$. Since m_3 is even, then by Lemma 2, we can change the labeling f at $v_{31}, v_{32}, \ldots, v_{3m_3}$ to be the labeling f' such that $|f'(v_{31}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$, the set of vertices labeling of f and the set of vertices labeling of f' are the same and the set of edge labeling of f and the set of edge labeling of f' are the same. Further, for any $i, 4 \leq i \leq n$ consider $|f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Since $m_i = 2m_{i-1}$, then by the way of labeling of T of Type I there exists $v^{''} \in \{v_{i1}, v_{i2}, \ldots, v_{im_i}\}$ such that $|f(v^{''}) - f(v)| = |f(v_{i1}) - f(v(i-1)m_{i-1})|$. Since m_i is even, then by Lemma 2, we can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling $f^{''}$ such that $|f^{''}(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ the set of vertices labeling of f and the set of vertices labeling of $f^{''}$ are the same and the set of edge labeling of f and the set of edge labeling of $f^{''}$ are the same. Hence T amit graceful labeling.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider in a same way as the **Case 1**, we get T admit graceful labeling.

We get the second case as the following theorem.

Theorem 2. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If $m_{i+1} = m_2 + m_3 + \ldots + m_i + 1, i = 2, 3, \ldots, n-1$, then T is a graceful labeling graph.*

Proof. Let T be the spider as shown in Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. Consider $|f(v_{31}) - f(v_{2m_2})|$. Since $m_3 = m_2 + 1$, then by the way of labeling of T of Type I we have $|f(v_{3m_3}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$. We can change the labeling f at $v_{31}, v_{32}, \ldots, v_{3m_3}$ to be the labeling f' such that $|f'(v_{31}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$ by reversing the order of theirs labels. Further, for any $i, 4 \leq i \leq n-1$ consider $|f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Since $m_i = m_2 + m_3 + \ldots + m_{i-1} + 1$, then by the way of labeling of T of Type I we have $|f(v_{im_i}) - f(v)| = |f(v_{i1}) - f(v(i-1)m_{i-1})|$. We can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling $f^{''}$ such that $|f^{''}(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ by reversing the order of theirs labels. Hence T amit graceful labeling.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider in a similar way as the **Case 1**, we get T admit graceful labeling.

Next we will extend Theorem 1 to be more general.

Theorem 3. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If $m_3, m_4, \ldots, m_n$ are even and $m_{i+1} \geq 2m_i$ for $i = 2, 3, \ldots, n-1$, then T is a graceful labeling graph.*

Proof. Let T be the spider as shown in Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. Consider $|f(v_{31}) - f(v_{2m_2})|$. Since $m_3 \geq 2m_2$, then by the way of labeling of T of Type I we have $|f(v_{3(m_2+1)}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$. Since m_3 is even, then by Lemma 2, we can change the labeling f at $v_{31}, v_{32}, \ldots, v_{3m_3}$ to be the labeling f' such that $|f'(v_{31}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$, the set of vertices labeling of f and the set of vertices labeling of f' are the same and the set of edge labeling of f and the set of edge labeling of f' are the same. Further, for any $i, 4 \leq i \leq n$ consider $|f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Let $m' = m_2 + m_3 + \ldots + m_{i-1}$. Since $m_i \geq 2m_{i-1} > m'$, then by the way of labeling of T of Type I we have $|f(v_{i(m'+1)}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Since m_i is even, then by Lemma 2, we can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling f'' such that $|f''(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ the set of vertices labeling of f and the set of vertices labeling of f'' are the same and the set of edge labeling of f and the set of edge labeling of f'' are the same. Hence T amit graceful labeling.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider similar way as the **Case 1** we get T admit graceful labeling.

We get the fourth case as the following theorem.

Theorem 4. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If $m_2 = m_3 = m_n$ and $m_{i+1} = m_2 + m_3 + \ldots + m_i, i = 3, 4, \ldots, n-2$, then T is a graceful labeling graph.*

Proof. Let T be the spider as shown in the Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. Consider $|f(v_{31}) - f(v_{2m_2})|$. Since $m_3 = m_2$, then by the way of labeling of T of Type I we have $|f(v_{41}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$. Further, for any $i, 4 \leq i \leq n-1$ consider $|f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Since $m_i = m_2 + m_3 + \ldots + m_{i-1}$, then by the way of labeling of T of Type I we have $|f(v_{(i+1)1}) - f(v)| = |f(v_{i1}) - f(v(i-1)m_{i-1})$. Next consider $|f(v_{n1}) - f(v_{(n-1)m_{n-1}})|$. Since $m_2 = m_3 = m_n$, then by the way of labeling of T of Type I we have $|f(v_{3m_3}) - f(v)| = |f(v_{n1}) - f(v_{(n-1)m_{n-1}})|$. We can change labeling f at $v_{31}, v_{32}, \ldots, v_{3m_3}$ to be the labeling f' such that $|f'(v_{31}) - f(v)| = |f(v_{n1}) - f(v_{(n-1)m_{n-1}})|$ by reversing the order of theirs labels. Hence T amit graceful labeling.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider in a similar way as the **Case 1**, we get T admit graceful labeling.

Next we will extend Theorem 2 to be more general.

Theorem 5. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If $m_3, m_4, \ldots, m_n$ are even and $m_{i+1} \geq m_2 + m_3 + \ldots + m_i + 1$ for $i = 2, 3, \ldots, n-1$, then T is a graceful labeling graph.*

Proof. Let T be the spider as shown in Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. Consider $|f(v_{31}) - f(v_{2m_2})|$. Since $m_3 \geq m_2 + 1$, then by the way of labeling of T of Type I we have $|f(v_{3(m_2+1)}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$. If $m_3 = m_2 + 1$, then we can change labeling f at $v_{31}, v_{32}, \ldots, v_{3m_3}$ to be the labeling f' such that $|f'(v_{31}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$ by reversing the order of theirs labels. Suppose that $m_3 > m_2 + 1$. Since m_3 is even, then by Lemma 2, we can change the labeling f at $v_{31}, v_{32}, \ldots, v_{3m_3}$ to be the labeling f'' such that $|f''(v_{31}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$, the set of vertices labeling of f and the set of vertices labeling of f'' are the same and the set of edge labeling of f and the set of edge labeling of $f^{''}$ are the same. Further, for any $i, 4 \leq i \leq n$ consider $|f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Let $m' = m_2 + m_3 + \ldots + m_{i-1}$. Since $m_i \geq m' + 1$, then by the way of labeling of T of Type I we have $|f(v_{m'+1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. If $m_i = m' + 1$, then we can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling $f^{'''}$ such that $|f^{'''}(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ by reversing the order of theirs labels. Suppose that $m_i > m'+1$. Since m_i is even, then by Lemma 2, we can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling f^{iv} such that $|f^{iv}(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ the set of vertices labeling of f and the set of vertices labeling of f^{iv} are the same and the set of edge labeling of f and the set of edge labeling of f^{iv} are the same. Hence T amit graceful labeling.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider in a similar way as the **Case 1**, we get T admit graceful labeling.

Next we will combine the results of Theorem 2 and Theorem 5 as the following theorem.

Theorem 6. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If T satisfies the following conditions:*
(i) if m_i is odd, then $m_i = m_2 + m_3 + \ldots + m_{i-1} + 1$,
(ii) if m_i is even, then $m_i \geq m_2 + m_3 + \ldots + m_{i-1} + 1$,
for any $i = 3, 4, \ldots, n$, then T is a graceful labeling graph.

Proof. Let T be the spider as shown in Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. For any $m_i, i = 3, 4, \ldots, n$ consider $|f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. Let $m' = m_2 + m_3 + \ldots + m_{i-1}$. Since $m_i \geq m'+1$, then by the way of labeling of T of Type I we have $|f(v_{m'+1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$. If m_i is odd,then by (i) we have $m_i = m' + 1$. We can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling f' such that $|f'(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ by reversing the order of theirs labels. If m_i is even, then by (ii) we have $m_i \geq m' + 1$. If $m_i = m' + 1$, then we can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling $f^{''}$ such that $|f^{''}(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ by reversing the order of theirs labels. Suppose that $m_i > m'+1$. Since m_i is even, then by Lemma 2, we can change labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ to be the labeling $f^{'''}$ such that $|f^{'''}(v_{i1}) - f(v)| = |f(v_{i1}) - f(v_{(i-1)m_{i-1}})|$ the set of vertices labeling of f and the set of vertices labeling of $f^{'''}$ are the same and the set of edge labeling of f and the set of edge labeling of $f^{'''}$ are the same. Hence T amit graceful labeling.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider in a similar way as the **Case 1**, we get T admit graceful labeling.

Next we will consider a spider graph in which all of its legs except one are equal.

Theorem 7. *Let T be a spider graph of n legs with lengths $m_1, m_2, m_3, \ldots, m_n$. If $m_2 = m_3 = \ldots = m_n$, then T is a graceful labeling graph.*

Proof. Let T be the spider as shown in Figure 1.

Case 1. If m_1 is odd, then let f be the labeling of T of Type I. Since $m_2 = m_3 = \ldots = m_n$, then by the way of labeling of T of Type I for n is odd we have $|f(v_{41}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$, $|f(v_{61}) - f(v)| = |f(v_{41}) - f(v_{3m_3})|$, ..., $|f(v_{(n-1)1}) - f(v)| = |f(v_{((n-1)/2+1)1}) - f(v_{((n-1)/2)m_{(n-1)/2}})|$ and $|f(v_{nm_n}) - f(v)| = |f(v_{((n-1)/2+2)1}) - f(v_{((n-1)/2+1)m_{(n-1)/2+1}})|$, $|f(v_{(n-2)m_{n-2}}) - f(v)| = |f(v_{((n-1)/2+3)1}) - f(v_{((n-1)/2+2)m_{(n-1)/2+2}})|$, ..., $|f(v_{3m_3}) - f(v)| = |f(v_{n1}) - f(v_{(n-1)m_{n-1}})|$. For each $i, i = 3, 5, \ldots, n$, if we change the labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ by reversing the order of theirs labels, then we get T is graceful. For n is even we have $|f(v_{41}) - f(v)| = |f(v_{31}) - f(v_{2m_2})|$, $|f(v_{61}) - f(v)| = |f(v_{41}) - f(v_{3m_3})|$, ..., $|f(v_{n1}) - f(v)| = |f(v_{(n/2+1)1}) - f(v_{(n/2)m_{n/2}})|$ and $|f(v_{(n-1)m_{n-1}}) - f(v)| = |f(v_{(n/2+2)1}) - f(v_{(n/2+1)m_{n/2+1}})|$, $|f(v_{(n-3)m_{n-3}}) - f(v)| = |f(v_{(n/2+3)1}) - f(v_{(n/2+2)m_{n/2+2}})|$, ..., $|f(v_{3m_3}) - f(v)| = |f(v_{n1}) - f(v_{(n-1)m_{n-1}})|$. For each $i, i = 3, 5, \ldots, n-1$, if we change the labeling f at $v_{i1}, v_{i2}, \ldots, v_{im_i}$ by reversing the order of theirs labels, then we get T is graceful.

Case 2. If m_1 is even, then let f be the labeling of T of Type II and consider similar as the **Case 1** we get T admit graceful labeling.

3. Conclusion

In this paper, we obtain that certain spider graphs, whose labeling is constructed in the forms of Type I or Type II, admit graceful labeling whenever the lengths of their legs satisfy the conditions presented in Theorems 1 through 7.

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References

[1] J. A. Gillian. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*, 16:DS6, 2015.

[2] G. S. Bloom and S. W. Golomb. Applications of numbered undirected graphs. *Proceedings of the IEEE*, 65(4):562–570, 1977.

[3] B. J. Septory, L. Susilowati, Dafik, V. Lokesha, and V. Nagamani. On the study of rainbow antimagic connection number of corona product of graphs. *European Journal of Pure and Applied Mathematics*, 16(1):271–285, 2023.

[4] B. J. Septory, L. Susilowati, Dafik, and V. Lokesha. On the study of rainbow antimagic connection number of comb product of friendship graph and tree. *Symmetry*, 15(1):42, 2023.

[5] G. Ringel. Theory of graphs and its applications. In *Proceedings of the Symposium Smolenice*, page 162, Prague, 1964. Czechoslovak Academy of Sciences.

[6] A. Rosa. On certain valuations of the vertices of a graph. In *Theory of Graphs, International Symposium, Rome, July 1966*, pages 349–355, New York, 1966. Gordon and Breach.

[7] C. Huang, A. Kotzig, and A. Rosa. Further results on tree labellings. *Utilitas Mathematica*, 21:31–48, 1982.

[8] P. Bahls, S. Lake, and A. Wertheim. Gracefulness of families of spiders. *Involve*, 3(3):241–247, 2010.

[9] P. Jampachon, K. Nakprasit, and T. Poomsa-ard. Graceful labeling of some classes of spider graphs with three legs greater than one. *Thai Journal of Mathematics*, 12(3):621–630, 2014.

[10] P. Jampachon and T. Poomsa-ard. Graceful labeling of spider graphs with three legs of lengths greater than one. *Far East Journal of Mathematical Sciences*, 100(1):51–64, 2016.

[11] A. Panpa and T. Poomsa-ard. On graceful spider graphs with at most four legs of length greater than one. *Journal of Applied Mathematics*, 2016:5327026, 2016.

[12] P. Hrnčiar and A. Haviar. All trees of diameter five are graceful. *Discrete Mathematics*, 233(1-3):133–150, 2001.

[13] K. Saengsura and T. Poomsa-ard. Graceful labeling of the classes of spider graphs with four legs of lengths greater than one $S_n(k, l, m, 2)$. *Far East Journal of Mathematical Sciences*, 100(2):255–269, 2016.