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Pythagorean Fuzzy HX-subgroups and Their Applications

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Abstract. In this paper, we introduce the notion of a pythagorean fuzzy HX-subgroup and a normal HX-subgroup. In addition, we prove various chracterisations for pythagorean fuzzy HX-subgroups and pythagorean normal HX-subgroups. Moreover, the notations of pythagorean fuzzy HX-subgroups homomorphisms and antihomomorphisms are introduced, and some related properties regarding the relationship between a pythagorean fuzzy set and its image are investigated. Characterisations of level pythagorean fuzzy HX-subgroups and normal HX-subgroups are proved. These results generalised some results regarding fuzzy HX-subgroups.

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1. Introduction

A generalisation of the classical set, the fuzzy set notion was first presented by Zadeh in 1965 [1]. This set addressed the relationship between elements and sets and answered the question: to what extent can this object belong to a particular set, as every object x has a value $\eta(x)$, where η is called a membership function $\eta : X \to [0.1]$.

Many ideas and abstractions have been expanded since fuzzy set theory's inception in order to effectively handle ambiguity and uncertainty [[2], [3], [4]]. After that, researchers found that the membership function is insufficient on its own to tackle some types of situations. This motivated Atanassov [5] and [6] to introduce the idea of intuitionistic fuzzy set by associating a fuzzy set non-membership with its membership function. The non-membership $\hat{\eta}$ and membership η in this class are satisfied: $0 \leq \eta(x) + \hat{\eta}(x) \leq 1$. This set can cope with ambiguous and unclear situations more effectively than fuzzy sets since it has both a non-membership and a membership functions, see [[7], [8], [9]].

However, if the situation required $\eta(x) + \hat{\eta}(x) \geq 1$, then intuitionistic fuzzy set theory is not applicable. To find a suitable answer in these situations, Yager [10] introduced the

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concept of pythagorean fuzzy subset. It assigns to every object x in a universe set two memberships: the membership $\eta(x)$ and the non-membership $\hat{\eta}(x)$ in which

$$0 \leq \eta(x)^2 + \hat{\eta}(x)^2 \leq 1.$$

. Consequently, a pythagorean fuzzy set could be thought of as an extension of an intuitionistic fuzzy set. Furthermore, because the condition $0 \leq \eta(x)^2 + \hat{\eta}(x)^2 \leq 1$ provides more pairs $(\eta, \hat{\eta})$ than the condition $0 \leq \eta(x) + \hat{\eta}(x) \leq 1$, this generalisation results in a greater number of applications that can be found using pythagorean fuzzy sets than those that are solved by intuitionistic fuzzy sets. Pythagorean fuzzy sets can therefore be used to solve more issues and produce precise and efficient algorithms [[11], [12], [7]]. The concept of pythagorean fuzzy sets was applied to groups, rings and modules, see [[13], [14], [15]].

Group theory is an important branch of mathematics. It can sort numerous problems in several fields of science. The applications of group theory described in many papers [[16], [17], [18]].

The idea of applying fuzzy settings on groups was introduced by Rosenfeld [19], who expanded on the idea of classical groups. After that, numerous efforts have been conducted to study fuzzy groups in several fuzzy environments.

The notation of HX-groups was introduced in [20], by Li Hongxing. The concept has since been the subject of various studies; see [21], [22], [23], [24], [25], [23].

Several authors have merged the ideas of HX groups with the concept of fuzzy sets to get some novel findings.

This study aims to establish the groundwork for a novel theory of pythagorean fuzzy HX-subgroup as it is the extension of fuzzy HX groups and intuitionistic fuzzy HX-subgroup.

In this paper, we introduce the notion of a pythagorean fuzzy HX-subgroups and normal HX-subgroups. In addition, we prove various chracterisations for pythagorean fuzzy HX-subgroups and pythagorean normal HX-subgroups. Then homomorphisms of pythagorean fuzzy HX-subgroups and antihomomorphisms of pythagorean fuzzy HX-subgroups are discussed. Several related properties regarding the relationship between a pythagorean fuzzy set and its image are investigated. Moreover, pythagorean fuzzy level HX-subgroups are discussed, and characterisations of these level pythagorean fuzzy HX-subgroups and normal HX-subgroups are presented.

Throughout this paper, we write PFSS to denote a pythagorean fuzzy subset, PF HX-SG to denote a pythagorean fuzzy HX-subgroup and PF HX-NSG to denote a pythagorean fuzzy HX-normal subgroup.

2. Pythagorean fuzzy HX-subgroups

Recall that [26] a non empty set $W \subseteq 2^G - \{\phi\}$ is called an HX group on G if W is a group with respect to algebraic operation defined by $MN = \{mn : m \in M, n \in N\}$ and the identity element is denoted by e. We present the following example:

Example 1. *Consider the multiplicative group $G = \{\pm 1, \pm i\}$. Then the set $W = \{\{1, -1\}, \{i, -i\}\}$ is an HX-group, where its identity is $\{1, -1\}$.*

This definition is applied to fuzzy settings in [27] and [28] and provide a definition for a fuzzy HX-SG, which defined as follows:

A fuzzy set Υ is a fuzzy HX-SG of an HX-group W if, for any $w_{01}, w_{02} \in W$, we have:

(1) $\Upsilon(w_{01}w_{02}) \geq \min\{\Upsilon(w_{01}), \Upsilon(w_{02})\}$.

(2) $\Upsilon(w_{01}^{-1}) = \Upsilon(w_{01})$.

Now, we are able to present the main definition of PF HX-SG.

Definition 1. *Let G be a group, $W \subseteq 2^G - \{\phi\}$ be an HX-group of G and $\Upsilon = \{(w; \bar{\Upsilon}(w), \hat{\Upsilon}(w)) : w \in W\}$ be a pythagorean fuzzy subset of W. Then Υ is called a PF HX-SG of W if:*

(1) $\bar{\Upsilon}^2(w_{01}w_{02}) \geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\}$.

(2) $\hat{\Upsilon}^2(w_{01}w_{02}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\}$.

(3) $\bar{\Upsilon}^2(w_{01}^{-1}) = \bar{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{01}^{-1}) = \hat{\Upsilon}^2(w_{01})$.

Example 2. *Consider the Klien 4-group $G = \{e, x, y, z\}$ and the XH-group $W = \{E, M\} = \{\{e, x\}, \{y, z\}\}$, such that*

$*$	E	M
E	E	M
M	M	E

Let η be a pythagorean fuzzy sets, where

$$\begin{aligned}
\bar{\eta}(e) = 0.6, \quad \hat{\eta}(e) = 0.3 \\
\bar{\eta}(x) = 0.5, \quad \hat{\eta}(x) = 0.5 \\
\bar{\eta}(y) = 0.4, \quad \hat{\eta}(y) = 0.3 \\
\bar{\eta}(z) = 0.3, \quad \hat{\eta}(z) = 0.5
\end{aligned}$$

Let $\bar{\Upsilon}(n) = \max\{\bar{\eta}(n) : n \in N \subseteq W\}$ and $\hat{\Upsilon}(n) = \min\{\hat{\eta}(n) : n \in N \subseteq W\}$. Thus

$$\begin{aligned}
\bar{\Upsilon}(E) &= \max\{\bar{\eta}(e), \bar{\eta}(x)\} = 0.6 \\
\hat{\Upsilon}(E) &= \min\{\hat{\eta}(e), \hat{\eta}(x)\} = 0.3 \\
\bar{\Upsilon}(M) &= \max\{\bar{\eta}(y), \bar{\eta}(z)\} = 0.4 \\
\hat{\Upsilon}(M) &= \min\{\hat{\eta}(y), \hat{\eta}(z)\} = 0.3
\end{aligned}$$

Thus Υ is a PF HX-SG.

Now, we inroduce the following proposition which will be used in later results.

Proposition 1. *Let Υ be a PF HX-SG of an HX-group W. Then*

$$\bar{\Upsilon}^2(e) \geq \bar{\Upsilon}^2(w_{01}),$$
$$\hat{\Upsilon}^2(e) \leq \hat{\Upsilon}^2(w_{01})$$

for all $w_{01} \in W$

Proof.

$$\bar{\Upsilon}^2(e) = \bar{\Upsilon}^2(w_{01}w_{01}^{-1}) \geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{01}^{-1})\} = \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{01})\} = \bar{\Upsilon}^2(w_{01})$$
$$\hat{\Upsilon}^2(e) = \hat{\Upsilon}^2(w_{01}w_{01}^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{01}^{-1})\} = \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{01})\} = \hat{\Upsilon}^2(w_{01}).$$

Using the above proposition, we are able to provide characterisation theorem of HX-SG.

Theorem 1. *Let W be an HX-group and Υ be a PF subset of W. Then Υ is a PF HX-SG of W if and only if*

$$\begin{aligned} \bar{\Upsilon}^2(w_{01}w_{02}^{-1}) &\geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\}, \\ \hat{\Upsilon}^2(w_{01}w_{02}^{-1}) &\leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\}. \end{aligned} \tag{1}$$

Proof. If Υ is a PF HX-SG of W, then

$$\bar{\Upsilon}^2(w_{01}w_{02}^{-1}) \geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02}^{-1})\} = \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\}$$

Also,$\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02}^{-1})\} = \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\}$.

If (1) holds, then

$$\bar{\Upsilon}^2(w_{01}^{-1}) = \bar{\Upsilon}^2(ew_{01}^{-1}) \geq \min\{\bar{\Upsilon}^2(e), \bar{\Upsilon}^2(w_{01})\} = \bar{\Upsilon}^2(w_{01}).$$

On the other hand,

$$\bar{\Upsilon}^2(w_{01}) = \bar{\Upsilon}^2(ew_{01}) \geq \min\{\bar{\Upsilon}^2(e), \bar{\Upsilon}^2(w_{01}^{-1})\} = \bar{\Upsilon}^2(w_{01}^{-1}).$$

Thus $\bar{\Upsilon}^2(w_{01}^{-1}) = \bar{\Upsilon}^2(w_{01})$. In addition,

$$\bar{\Upsilon}^2(w_{01}w_{02}) = \bar{\Upsilon}^2(w_{01}(w_{02}^{-1})^{-1}) \geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02}^{-1})\} = \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\}.$$

Similarly,

$$\hat{\Upsilon}^2(w_{01}^{-1}) = \hat{\Upsilon}^2(ew_{01}^{-1}) \leq \max\{\hat{\Upsilon}^2(e), \hat{\Upsilon}^2(w_{01})\} = \hat{\Upsilon}^2(w_{01})$$

and

$$\hat{\Upsilon}^2(w_{01}) = \hat{\Upsilon}^2(ew_{01}) \leq \max\{\hat{\Upsilon}^2(e), \hat{\Upsilon}^2(w_{01}^{-1})\} = \hat{\Upsilon}^2(w_{01}^{-1}).$$

Thus $\hat{\Upsilon}^2(w_{01}^{-1}) = \hat{\Upsilon}^2(w_{01})$. In addition,

$$\hat{\Upsilon}^2(w_{01}w_{02}) = \Upsilon^2(w_{01}(w_{02}^{-1})^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02}^{-1})\} = \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\}$$

Hence Υ is a PF HX-SG of W.

Proposition 2. *Let W be an HX-group and Υ_i be PF HX-SGs of W. Then $\bigcap_i \Upsilon_i$ is a PF HX-SG of W.*

Proof. Clear.

We define two well known pytharorean fuzzy subsets:

Definition 2. *Let W be an HX-group and P be a PFSS of W. Then*

(1) $P^\star = \bar{\Upsilon}_P^\star \cap \hat{\Upsilon}_P^\star$, *where*

$$\bar{\Upsilon}_P^\star = \{w \in W : \bar{\Upsilon^2}_P(w) > 0\}$$
$$\hat{\Upsilon}_P^\star = \{w \in W : \hat{\Upsilon}^2(w) < 1\}$$

(2) $P_\star = \bar{\Upsilon}_{\star P} \cap \hat{\Upsilon}_{\star P}$, *where*

$$\bar{\Upsilon}_{\star P} = \{w \in W : \bar{\Upsilon^2}_P(w) = 1\}$$
$$\hat{\Upsilon}_{\star P} = \{w \in W : \hat{\Upsilon}^2(w) = 0\}$$

Now, we prove that the above PFSSs are subgroups of G in the case that Υ_P is a PF HX-SG.

Theorem 2. *If Υ_P is a PF HX-SG of G, then $P^\star$ is a subgroup of G.*

Proof. Suppose that $w_{01}, w_{02} \in P^\star$. Then $\bar{\Upsilon^2}(w_{01}) > 0, \bar{\Upsilon^2}(w_{02}) > 0$ and $\hat{\Upsilon}^2(w_{01}) < 1, \hat{\Upsilon}^2(w_{02}) < 1$. By hypothesis, $\bar{\Upsilon^2}(w_{01}w_{02}^{-1}) \geq \min\{\bar{\Upsilon^2}(w_{01}), \bar{\Upsilon^2}(w_{02})\} > 0$. Also, $\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\} < 1$. Hence $w_{01}w_{02}^{-1} \in P^\star$ and therefore, $P^\star$ is a subgroup of G.

Theorem 3. *If Υ_P is a PF HX-SG of G, then $P_\star$ is a subgroup of G.*

Proof. Assume that $w_{01}, w_{02} \in P^\star$. Then $\bar{\Upsilon^2}(w_{01}) = 1, \bar{\Upsilon^2}(w_{02}) = 1$ and $\hat{\Upsilon}^2(w_{01}) = 0, \hat{\Upsilon}^2(w_{02}) = 0$. By hypothesis, $\bar{\Upsilon^2}(w_{01}w_{02}^{-1}) \geq \min\{\bar{\Upsilon^2}(w_{01}), \bar{\Upsilon^2}(w_{02})\} = 1$. Also, $\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\} = 0$. Hence $w_{01}w_{02}^{-1} \in P_\star$ and therefore, $P_\star$ is a subgroup of G.

A (ζ, δ)-level pythagorean fuzzy subset can be defined as follows:

Definition 3. *Let Υ be a PFSS of W. Then*

$$\Upsilon_{(\zeta,\delta)} = \{w \in W : \bar{\Upsilon^2}(w) \geq \zeta \text{ and } \hat{\Upsilon}^2(w) \leq \delta\},$$

where $\zeta, \delta \in [0, 1]$.

An essential question arises: is there a relationship between these level PFSSs $\Upsilon_{(\zeta,\delta)}$ and a PFSS Υ? We present the following theorem to answer this question.

Theorem 4. *Let Υ be a PFSS of W. Then $\Upsilon_{(\zeta,\delta)}$ is an HX-SG, for all $\zeta, \delta \in [0, 1]$, if and only if Υ is a PF HX-SG.*

Proof. Suppose that $\Upsilon_{(\zeta,\delta)}$ is an HX-SG. Assume that there exists $w_{01}, w_{02} \in W$ such that $\bar{\Upsilon}^2(w_{01}w_{02}^{-1}) < \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\}$. Then $\bar{\Upsilon}^2(w_{01}w_{02}^{-1}) < \zeta$, for some $\zeta \in [0,1]$ and this contradicts that $\Upsilon_{(\zeta,\delta)}$ is an HX-SG. Thus $\bar{\Upsilon}^2(w_{01}w_{02}^{-1}) \geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\}$. Now, let $\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) > \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\}$. This implies that $\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) > \delta$, for some $\delta \in [0,1]$ and this contradicts that $\Upsilon_{(\zeta,\delta)}$ is an HX-SG. This menas that $\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\}$. Hence Υ is a PF HX-SG.

Conversely, suppose that Υ is a PF HX-SG. Let $w_{01}, w_{02} \in \Upsilon_{(\zeta,\delta)}$, for some (ζ,δ). Then $\bar{\Upsilon}^2(w_{01}) \geq \zeta$, $\bar{\Upsilon}^2(w_{02}) \geq \zeta$, $\hat{\Upsilon}^2(w_{01}) \leq \delta$ and $\hat{\Upsilon}^2(w_{02}) \leq \delta$. That Υ is a PF HX-SG implies that $\bar{\Upsilon}^2(w_{01}w_{02}^{-1}) \geq \min\{\bar{\Upsilon}^2(w_{01}), \bar{\Upsilon}^2(w_{02})\} \geq \zeta$ and $\hat{\Upsilon}^2(w_{01}w_{02}^{-1}) \leq \max\{\hat{\Upsilon}^2(w_{01}), \hat{\Upsilon}^2(w_{02})\} \leq \delta$. Hence $w_{01}w_{02}^{-1} \in \Upsilon_{(\zeta,\delta)}$ and therefore, $\Upsilon_{(\zeta,\delta)}$ is an HX-SG.

3. Pythagorean fuzzy HX-normal subgroups

In this section, we study PF HX-NSGs. We first introduce the definition of left cosets:

Definition 4. *Let Υ be an HX-SG of W. Then a left coset of Υ in W is defined by $w_{01}\Upsilon(w_{02}) = \Upsilon(w_{01}^{-1}w_{02})$ for all $w_{01}, w_{02} \in W$, that is $w_{01}\bar{\Upsilon}^2(w_{02}) = \bar{\Upsilon}^2(w_{01}^{-1}w_{02})$ and $w_{01}\hat{\Upsilon}^2(w_{02}) = \hat{\Upsilon}^2(w_{01}^{-1}w_{02})$ for all $w_{01}, w_{02} \in W$.*

In order to explain the above definition, we present the following example:

Example 3. *Consider the group $(\mathbb{Z}_7^*, \cdot_7)$ and the XH-group $W = \{E, M, N\} = \{\{1,6\},\{2,5\},\{3,4\}\}$, such that*

$*$	E	M	N
E	E	M	N
M	M	N	E
N	N	E	M

Let η be a pythagorean fuzzy set, where

$$\begin{aligned}
\bar{\eta}(1) = 0.6, &\quad \hat{\eta}(1) = 0.7\\
\bar{\eta}(2) = 0.5, &\quad \hat{\eta}(2) = 0.6\\
\bar{\eta}(3) = 0.3, &\quad \hat{\eta}(3) = 0.5\\
\bar{\eta}(4) = 0.5, &\quad \hat{\eta}(4) = 0.4\\
\bar{\eta}(5) = 0.4, &\quad \hat{\eta}(5) = 0.4\\
\bar{\eta}(6) = 0.5, &\quad \hat{\eta}(6) = 0.5
\end{aligned}$$

Let $\bar{\Upsilon}(n) = \max\{\bar{\eta}(n) : n \in N \subseteq W\}$ and $\hat{\Upsilon}(n) = \min\{\hat{\eta}(n) : n \in N \subseteq W\}$. Thus

$$\bar{\Upsilon}(E) = \max\{\bar{\eta}(1), \bar{\eta}(6)\} = 0.6$$

$$\hat{\Upsilon}(E) = \min\{\hat{\eta}(1), \hat{\eta}(6)\} = 0.5$$
$$\bar{\Upsilon}(M) = \max\{\bar{\eta}(2), \bar{\eta}(5)\} = 0.5$$
$$\hat{\Upsilon}(M) = \min\{\hat{\eta}(2), \hat{\eta}(5)\} = 0.4$$
$$\bar{\Upsilon}(N) = \max\{\bar{\eta}(3), \bar{\eta}(4)\} = 0.5$$
$$\hat{\Upsilon}(N) = \min\{\hat{\eta}(3), \hat{\eta}(4)\} = 0.4$$

We find the left coset $M\Upsilon(N)$. By definition, $M\Upsilon(N) = \Upsilon(M^{-1}N)$. That is:

$$M\bar{\Upsilon}^2(N) = \bar{\Upsilon}^2(M^{-1}N) = \bar{\Upsilon}^2(NN) = \bar{\Upsilon}^2(M) = 0.36$$
$$M\hat{\Upsilon}^2(N) = \hat{\Upsilon}^2(M^{-1}N) = \hat{\Upsilon}^2(NN) = \hat{\Upsilon}^2(M) = 0.16.$$

Similarly, we can compute any left coset.

Now, we are ready to define a PF HX-NSG.

Definition 5. *Let G be a group, $W \subseteq 2^G - \{\phi\}$ be an HX-group of G and $\Upsilon = \{(w; \bar{\Upsilon}(w), \hat{\Upsilon}(w)) : w \in W\}$ be a PF HX-SG of W. Then Υ is called a PF HX-NSG of W if: $\bar{\Upsilon}^2(w_{01}w_{02}) = \bar{\Upsilon}^2(w_{02}w_{01})$ and $\hat{\Upsilon}^2(w_{01}w_{02}) = \hat{\Upsilon}^2(w_{02}w_{01})$. Alternatively, Υ is a PF HX-NSG of W if $w_{01}\Upsilon(w_{02}) = \Upsilon(w_{02})w_{01}$ for all $w_{01}, w_{02} \in W$.*

Example 4. *Consider W, η and Υ as in example 3. Then:*

$$\bar{\Upsilon}^2(EM) = \bar{\Upsilon}^2(ME) = 0.25$$
$$\hat{\Upsilon}^2(EM) = \hat{\Upsilon}^2(ME) = 0.16$$
$$\bar{\Upsilon}^2(EN) = \bar{\Upsilon}^2(NE) = 0.25$$
$$\hat{\Upsilon}^2(EN) = \hat{\Upsilon}^2(NE) = 0.16$$
$$\bar{\Upsilon}^2(NM) = \bar{\Upsilon}^2(MN) = 0.36$$
$$\hat{\Upsilon}^2(NM) = \hat{\Upsilon}^2(MN) = 0.25$$

Thus Υ is a PF HXN-SG.

Turning to PF HX-NSGs, we can say more about the properties of them. The following theorems described these properties:

Theorem 5. *Let W be an HX-group and Υ be a PF HX-SG of W. Then Υ is a PF HX-NSG of W if and only if*

$$\begin{aligned} \bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) &= \bar{\Upsilon}^2(w_{02}), \\ \hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) &= \hat{\Upsilon}^2(w_{02}). \end{aligned} \tag{2}$$

Proof. If Υ is a PF HX-NSG of W, then

$$\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \bar{\Upsilon}^2((w_{01}^{-1}w_{02})w_{01})$$

$$
\begin{aligned}
&= \bar{\Upsilon}^2((w_{02}w_{01}^{-1})w_{01}) \\
&= \bar{\Upsilon}^2(w_{02}(w_{01}^{-1}w_{01})) \\
&= \bar{\Upsilon}^2(w_{02})
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) &= \hat{\Upsilon}^2((w_{01}^{-1}w_{02})w_{01}) \\
&= \hat{\Upsilon}^2((w_{02}w_{01}^{-1})w_{01}) \\
&= \hat{\Upsilon}^2(w_{02}(w_{01}^{-1}w_{01})) \\
&= \hat{\Upsilon}^2(w_{02})
\end{aligned}
$$

Conversely, if (2) holds, then

$$
\begin{aligned}
\bar{\Upsilon}^2(w_{01}w_{02}) &= w_{01}^{-1}\bar{\Upsilon}^2(w_{02}) \\
&= w_{01}^{-1}\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) \\
&= \bar{\Upsilon}^2(w_{01}w_{01}^{-1}w_{02}w_{01}) \\
&= \bar{\Upsilon}^2(w_{02}w_{01})
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\Upsilon}^2(w_{01}w_{02}) &= w_{01}^{-1}\hat{\Upsilon}^2(w_{02}) \\
&= w_{01}^{-1}\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) \\
&= \hat{\Upsilon}^2(w_{01}w_{01}^{-1}w_{02}w_{01}) \\
&= \hat{\Upsilon}^2(w_{02}w_{01})
\end{aligned}
$$

Hence Υ is a PF HX-NSG.

Proposition 3. *Let W be an HX-group and Υ_i be PF HX-NSGs of W. Then $\bigcap_i \Upsilon_i$ is a PF HX-NSG of W.*

Proof. Clear.

The following two theorems are the analogue of theorem 2 and theorem 3:

Theorem 6. *If Υ_P is a PF HX-NSG of G, then $P^\star$ is a normal subgroup of G.*

Proof. Since Υ_P is a PF HX-NSG of G, then Υ_P is a PF HX-SG of G, thus, by theorem 2, $P^\star$ is a subgroup of G. Now, suppose that $w_{01}, w_{02} \in P^\star$. Then

$$\bar{\Upsilon}^2(w_{01}) > 0, \bar{\Upsilon}^2(w_{02}) > 0 \quad \text{and} \quad \hat{\Upsilon}^2(w_{01}) < 1, \hat{\Upsilon}^2(w_{02}) < 1.$$

That Υ_P is a PF HX-NSG implies that

$$\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \bar{\Upsilon}^2(w_{02})\} > 0 \quad \text{and} \quad \hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \hat{\Upsilon}^2(w_{02})\} < 1.$$

Hence $w_{01}^{-1}w_{02}w_{01} \in P^\star$ and therefore, $P^\star$ is a normal subgroup of G.

Theorem 7. *If Υ_P is a PF HX-NSG of G, then $P_\star$ is a normal subgroup of G.*

Proof. Since Υ_P is a PF HX-NSG of G, then Υ_P is a PF HX-SG of G. Thus, by theorem 3, $P_\star$ is a subgroup of G. Now, let $w_{01}, w_{02} \in P_\star$. Then $\bar{\Upsilon}^2(w_{01}) = 1, \bar{\Upsilon}^2(w_{02}) = 1$ and $\hat{\Upsilon}^2(w_{01}) = 0, \hat{\Upsilon}^2(w_{02}) = 0$. That Υ_P is a PF HX-NSG implies that $\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \bar{\Upsilon}^2(w_{02})\} = 1$ and $\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \hat{\Upsilon}^2(w_{02})\} = 0$. Hence $w_{01}^{-1}w_{02}w_{01} \in P_\star$ and therefore, $P^\star$ is a normal subgroup of G.

For (ζ, δ)-level PFSSs, we prove the following theorem:

Theorem 8. *Let Υ be a PFSS of W. Then $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG if and only if Υ is a PF HX-NSG.*

Proof. Suppose that $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG. Then $\Upsilon_{(\zeta,\delta)}$ is a PF HX-SG and hence by theorem 4, Υ is a PF HX-SG. Assume that there exists $w_{01}, w_{02} \in W$ such that $\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) < \bar{\Upsilon}^2(w_{02}) = \zeta$, for some $\zeta \in [0,1]$. Then $\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01} < \zeta$ and this contradicts that $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG. In the case that $\bar{\Upsilon}^2(w_{02}) < \bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01})$, then $\bar{\Upsilon}^2(w_{02}) < \bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \zeta$, for some $\zeta \in [0,1]$ and this contradicts that $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG. Thus $\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \bar{\Upsilon}^2(w_{02})$. Now, let $\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) > \hat{\Upsilon}^2(w_{02}) = \delta$, for some $\delta \in [0,1]$. Then $\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) > \delta$ and this contradicts that $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG. In the case that $\hat{\Upsilon}^2(w_{02}) > \hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01})$, then $\hat{\Upsilon}^2(w_{02}) > \hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \zeta$, for some $\zeta \in [0,1]$ and this contradicts that $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG. Thus $\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \hat{\Upsilon}^2(w_{02})$. Thus $\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \hat{\Upsilon}^2(w_{02})$. Therefore, Υ is a PF HX-NSG.

Conversely, assume that Υ is a PF HX-NSG. Let $w_{01}, w_{02} \in \Upsilon_{(\zeta,\delta)}$, for some (ζ, δ). Then $\bar{\Upsilon}^2(w_{01}) \geq \zeta$, $\bar{\Upsilon}^2(w_{02}) \geq \zeta$, $\hat{\Upsilon}^2(w_{01}) \leq \delta$ and $\hat{\Upsilon}^2(w_{02}) \leq \delta$. That Υ is a PF HX-NSG implies that $\bar{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \bar{\Upsilon}^2(w_{02})\} \geq \zeta$ and $\hat{\Upsilon}^2(w_{01}^{-1}w_{02}w_{01}) = \hat{\Upsilon}^2(w_{02}) \leq \delta$. Hence $w_{01}^{-1}w_{02}w_{01} \in \Upsilon_{(\zeta,\delta)}$ and therefore, $\Upsilon_{(\zeta,\delta)}$ is a PF HX-NSG.

4. Pythagorean fuzzy homomorphism of HX-subgroups

Let P, S be two HX-groups, L a PF HX-SG of P and N a PF-HX-SG of S. Consider a homomorphism

$$\tau : P \longrightarrow S$$

For $s \in S$, we define:

$$\eta_{\tau(L)}(s) = \begin{cases} \max\{\bar{\Upsilon}_L(p) : s = \tau(p)\} & \text{if } s \in \mathrm{Im}(\tau) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\eta}_{\tau(L)}(s) = \begin{cases} \min\{\hat{\Upsilon}_L(p) : s = \tau(p)\} & \text{if } s \in \mathrm{Im}(\tau) \\ 1 & \text{otherwise} \end{cases}$$

After presenting the definition of PF homomorphism, we are now ready to provide a relationship between a PFSS and its image.

Theorem 9. *Let $\tau : P \longrightarrow S$ be a homomorphism of HX-groups. If Υ is a PF-HX-SG of P, then $\tau(\Upsilon)$ is a PF-HX-SG of S.*

Proof. Suppose that $\tau(\Upsilon) = \{(\tau(p), \eta, \hat{\eta}) : \tau(p) \in S\}$. Let $\tau(p_{01}), \tau(p_{01}) \in S$. Then

$$\begin{aligned}\eta^2(\tau(p_{01})(\tau(p_{02}))^{-1}) &= \eta^2(\tau(p_{01})\tau(p_{02}^{-1}))\\ &= \eta^2(\tau(p_{01}p_{02}^{-1})\\ &\geq \bar{\Upsilon}^2(p_{01}p_{02}^{-1})\\ &\geq \min\{\bar{\Upsilon}^2(p_{01}, \bar{\Upsilon}^2(p_{02})\}\\ &= \min\{\eta^2(\tau(p_{01})), \eta^2(\tau(p_{02}))\}\end{aligned}$$

Also,

$$\begin{aligned}\hat{\eta^2}(\tau(p_{01})(\tau(p_{02}))^{-1}) &= \hat{\eta^2}(\tau(p_{01})\tau(p_{02}^{-1}))\\ &= \hat{\eta^2}(\tau(p_{01}p_{02}^{-1})\\ &\leq \hat{\Upsilon}^2(p_{01}p_{02}^{-1})\\ &\leq \max\{\hat{\Upsilon}^2(p_{01}, \hat{\Upsilon}^2(p_{02})\}\\ &= \max\{\hat{\eta^2}(\tau(p_{01})), \hat{\eta^2}(\tau(p_{02}))\}\end{aligned}$$

Hence $\tau(\Upsilon)$ is a PF-HX-SG of S.

Theorem 10. *Let $\tau : P \longrightarrow S$ be a homomorphism of HX-groups. If Υ is a PF-HX-NSG of P, then $\tau(\Upsilon)$ is a PF HX-NSG of S.*

Proof. Suppose that $\tau(\Upsilon) = \{(\tau(p), \eta, \hat{\eta}) : \tau(p) \in S\}$. Since Υ is a PF-HX-NSG of P, then it is a PF-HX-SG and, by the previous theorem, $\tau(\Upsilon)$ is a PF-HX-SG of S. We need to prove that it is normal. Let $\tau(p_{01}), \tau(p_{01}) \in S$. Then

$$\begin{aligned}\eta^2(\tau(p_{01})\tau(p_{02})) &= (\tau(p_{01}))^{-1}\eta^2(\tau(p_{02}))\\ &= (\tau(p_{01}))^{-1}\bar{\Upsilon}^2((p_{02})\\ &= (\tau(p_{01}))^{-1}\bar{\Upsilon}^2((p_{01})^{-1}p_{02}p_{01})\\ &\leq (\tau(p_{01}))^{-1}\eta^2(\tau((p_{01})^{-1}p_{02}p_{01}))\\ &= \eta^2(\tau(p_{01})\tau((p_{01})^{-1}p_{02}p_{01}))\\ &= \eta^2(\tau(p_{01}(p_{01})^{-1}p_{02}p_{01}))\\ &= \eta^2(\tau(p_{02}p_{01}))\\ &= \eta^2(\tau(p_{02}\tau(p_{01})))\end{aligned}$$

Thus $\eta^2(\tau(p_{01})\tau(p_{02})) \leq \eta^2(\tau(p_{02}\tau(p_{01})))$. On the other hand,

$$\eta^2(\tau(p_{02})\tau(p_{01})) = (\tau(p_{02}))^{-1}\eta^2(\tau(p_{01})$$

$$
\begin{aligned}
&= (\tau(p_{02}))^{-1}\bar{\Upsilon}^2((p_{01}) \\
&= (\tau(p_{02}))^{-1}\bar{\Upsilon}^2((p_{02})^{-1}p_{01}p_{02}) \\
&\leq (\tau(p_{02}))^{-1}\eta^2(\tau((p_{02})^{-1}p_{01}p_{02})) \\
&= \eta^2(\tau(p_{02})\tau((p_{02})^{-1}p_{01}p_{02})) \\
&= \eta^2(\tau(p_{02}(p_{02})^{-1}p_{01}p_{02})) \\
&= \eta^2(\tau(p_{01}p_{02})) \\
&= \eta^2(\tau(p_{01}\tau(p_{02}))).
\end{aligned}
$$

Then $\eta^2(\tau(p_{02})\tau(p_{01})) \leq \eta^2(\tau(p_{01}\tau(p_{02})))$. Hence $\eta^2(\tau(p_{01}\tau(p_{02}))) = \eta^2(\tau(p_{02}\tau(p_{01})))$.

In addition,

$$
\begin{aligned}
\hat{\eta}^2(\tau(p_{01})\tau(p_{02})) &= (\tau(p_{01}))^{-1}\hat{\eta}^2(\tau(p_{02})) \\
&= (\tau(p_{01}))^{-1}\hat{\Upsilon}^2((p_{02}) \\
&= (\tau(p_{01}))^{-1}\hat{\Upsilon}^2((p_{01})^{-1}p_{02}p_{01}) \\
&\geq (\tau(p_{01}))^{-1}\hat{\eta}^2(\tau((p_{01})^{-1}p_{02}p_{01})) \\
&= \hat{\eta}^2(\tau(p_{01})[]\tau((p_{01})^{-1}p_{02}p_{01})) \\
&= \hat{\eta}^2(\tau(p_{01}(p_{01})^{-1}p_{02}p_{01})) \\
&= \hat{\eta}^2(\tau(p_{02}p_{01})) \\
&= \hat{\eta}^2(\tau(p_{02}\tau(p_{01})))
\end{aligned}
$$

This implies that $\hat{\eta}^2(\tau(p_{01})\tau(p_{02})) \geq \hat{\eta}^2(\tau(p_{02}\tau(p_{01})))$. On the other hand,

$$
\begin{aligned}
\hat{\eta}^2(\tau(p_{02})\tau(p_{01})) &= (\tau(p_{02}))^{-1}\hat{\eta}^2(\tau(p_{01}) \\
&= (\tau(p_{02}))^{-1}\hat{\Upsilon}^2((p_{01}) \\
&= (\tau(p_{02}))^{-1}\hat{\Upsilon}^2((p_{02})^{-1}p_{01}p_{02}) \\
&\geq (\tau(p_{02}))^{-1}\hat{\eta}^2(\tau((p_{02})^{-1}p_{01}p_{02})) \\
&= \hat{\eta}^2(\tau(p_{02})\tau((p_{02})^{-1}p_{01}p_{02})) \\
&= \hat{\eta}^2(\tau(p_{02}(p_{02})^{-1}p_{01}p_{02})) \\
&= \hat{\eta}^2(\tau(p_{01}p_{02})) \\
&= \hat{\eta}^2(\tau(p_{01}\tau(p_{02})))
\end{aligned}
$$

Then $\hat{\eta}^2(\tau(p_{02})\tau(p_{01})) \geq \hat{\eta}^2(\tau(p_{01}\tau(p_{02})))$. Hence $\hat{\eta}^2(\tau(p_{01}\tau(p_{02}))) = \hat{\eta}^2(\tau(p_{02}\tau(p_{01})))$. Therefore, $\tau(\Upsilon)$ is a PF-NHX-SG of S.

Theorem 11. *Let $\tau : P \longrightarrow S$ be an antihomomorphism of HX-groups. If Υ is a PF-HX-SG of P, then $\tau(\Upsilon)$ is a PF-HX-SG of S.*

Proof. Suppose that $\tau(\Upsilon) = \{(\tau(p), \eta, \hat{\eta}) : \tau(p) \in S\}$. Let $\tau(p_{01}), \tau(p_{01}) \in S$. Then

$$\begin{aligned}
\eta^2(\tau(p_{01})(\tau(p_{02}))^{-1}) &= \eta^2(\tau(p_{01})\tau(p_{02}^{-1})) \\
&= \eta^2(\tau(p_{02}^{-1}p_{01}) \\
&\geq \bar{\Upsilon}^2(p_{02}^{-1}p_{01}) \\
&\geq \min\{\bar{\Upsilon}^2(p_{02})^{-1}, \bar{\Upsilon}^2(p_{01})\} \\
&= \min\{\bar{\Upsilon}^2(p_{01}), \bar{\Upsilon}^2(p_{02})\} \\
&= \min\{\eta^2(\tau(p_{01})), \eta^2(\tau(p_{02}))\}
\end{aligned}$$

Also,

$$\begin{aligned}
\hat{\eta}^2(\tau(p_{01})(\tau(p_{02}))^{-1}) &= \hat{\eta}^2(\tau(p_{01})\tau(p_{02}^{-1})) \\
&= \hat{\eta}^2(\tau(p_{02}^{-1}p_{01}) \\
&\leq \hat{\Upsilon}^2(p_{02}^{-1}p_{01}) \\
&\leq \max\{\hat{\Upsilon}^2(p_{02}^{-1}, \hat{\Upsilon}^2(p_{01})\} \\
&= \max\{\hat{\Upsilon}^2(p_{01}, \hat{\Upsilon}^2(p_{02})\} \\
&= \max\{\hat{\eta}^2(\tau(p_{01})), \hat{\eta}^2(\tau(p_{02}))\}.
\end{aligned}$$

Hence $\tau(\Upsilon)$ is a PF-HX-SG of S.

Theorem 12. *Let $\tau : P \longrightarrow S$ be an antihomomorphism of HX-groups. If Υ is a PF-HX-NSG of P, then $\tau(\Upsilon)$ is a PF-NHX-SG of S.*

Proof. Suppose that $\tau(\Upsilon) = \{(\tau(p), \eta, \hat{\eta}) : \tau(p) \in S\}$. Since Υ is a PF-HX-NSG of P, then it is a PF-HX-SG and, by the previous theorem, $\tau(\Upsilon)$ is a PF-HX-SG of S. We need to prove that it is normal. Let $\tau(p_{01}), \tau(p_{01}) \in S$. Then

$$\begin{aligned}
\eta^2(\tau(p_{01})\tau(p_{02})) &= (\tau(p_{01}))^{-1}\eta^2(\tau(p_{02})) \\
&= (\tau(p_{01}))^{-1}\bar{\Upsilon}^2((p_{02}) \\
&= (\tau(p_{01}))^{-1}\bar{\Upsilon}^2(p_{01}p_{02}(p_{01})^{-1}) \\
&\leq (\tau(p_{01}))^{-1}\eta^2(\tau(p_{01}p_{02}(p_{01})^{-1})) \\
&= \eta^2(\tau(p_{01})\tau(p_{01}p_{02}(p_{01})^{-1})) \\
&= \eta^2(\tau(p_{01}p_{02}(p_{01})^{-1}p_{01})) \\
&= \eta^2(\tau(p_{01}p_{02})) \\
&= \eta^2(\tau(p_{02})\tau(p_{01})))
\end{aligned}$$

Then $\eta^2(\tau(p_{01}\tau(p_{02}))) \leq \eta^2(\tau(p_{02}\tau(p_{01})))$. On the other hand,

$$\eta^2(\tau(p_{02})\tau(p_{01})) = (\tau(p_{02}))^{-1}\eta^2(\tau(p_{01}))$$

$$
\begin{aligned}
&= (\tau(p_{02}))^{-1}\bar{\Upsilon}^2((p_{01}) \\
&= (\tau(p_{02}))^{-1}\bar{\Upsilon}^2(p_{02}p_{01}(p_{02})^{-1}) \\
&\leq (\tau(p_{02}))^{-1}\eta^2(\tau(p_{02}p_{01}(p_{02})^{-1})) \\
&= \eta^2(\tau(p_{02})\tau(p_{02}p_{01}(p_{02})^{-1})) \\
&= \eta^2(\tau(p_{02}p_{01}(p_{02})^{-1}p_{02})) \\
&= \eta^2(\tau(p_{02}p_{01})) \\
&= \eta^2(\tau(p_{01})\tau(p_{02})))
\end{aligned}
$$

This impliea that $\eta^2(\tau(p_{02}\tau(p_{02}))) \leq \eta^2(\tau(p_{01}\tau(p_{02})))$. Thus $\eta^2(\tau(p_{01}\tau(p_{02}))) = \eta^2(\tau(p_{02}\tau(p_{01})))$. In addition,

$$
\begin{aligned}
\hat{\eta}^2(\tau(p_{01})\tau(p_{02})) &= (\tau(p_{01}))^{-1}\hat{\eta}^2(\tau(p_{02})) \\
&= (\tau(p_{01}))^{-1}\hat{\Upsilon}^2((p_{02}) \\
&= (\tau(p_{01}))^{-1}\hat{\Upsilon}^2(p_{01}p_{02}(p_{01})^{-1}) \\
&\leq (\tau(p_{01}))^{-1}\hat{\eta}^2(\tau(p_{01}p_{02}(p_{01})^{-1})) \\
&= \hat{\eta}^2(\tau(p_{01})\tau(p_{01}p_{02}(p_{01})^{-1})) \\
&= \hat{\eta}^2(\tau(p_{01}p_{02}(p_{01})^{-1}p_{01})) \\
&= \hat{\eta}^2(\tau(p_{01}p_{02})) \\
&= \hat{\eta}^2(\tau(p_{02})\tau(p_{01})))
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\hat{\eta}^2(\tau(p_{02})\tau(p_{01})) &= (\tau(p_{02}))^{-1}\hat{\eta}^2(\tau(p_{01})) \\
&= (\tau(p_{02}))^{-1}\hat{\Upsilon}^2((p_{01}) \\
&= (\tau(p_{02}))^{-1}\hat{\Upsilon}^2(p_{02}p_{01}(p_{02})^{-1}) \\
&\leq (\tau(p_{02}))^{-1}\hat{\eta}^2(\tau(p_{02}p_{01}(p_{02})^{-1})) \\
&= \hat{\eta}^2(\tau(p_{02})\tau(p_{02}p_{01}(p_{02})^{-1})) \\
&= \hat{\eta}^2(\tau(p_{02}p_{01}(p_{02})^{-1}p_{02})) \\
&= \hat{\eta}^2(\tau(p_{02}p_{01})) \\
&= \hat{\eta}^2(\tau(p_{01})\tau(p_{02})))
\end{aligned}
$$

Thus $\hat{\eta}^2(\tau(p_{01}\tau(p_{02}))) = \hat{\eta}^2(\tau(p_{02}\tau(p_{01})))$. Therefore, $\tau(\Upsilon)$ is a PF-NHX-SG of S.

Theorem 13. *Let $\tau : P \longrightarrow S$ be a homomorphism of HX-groups and let Υ^2 be a PFSS of W. If $\Upsilon_{(\zeta,\delta)}$ is an HX-SG, then $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-SG.*

Proof. Suppose that $\tau(\Upsilon_{(\zeta,\delta)}) = \{(\tau(p), \eta^2, \hat{\eta^2}) : p \in \Upsilon_{(\zeta,\delta)}\}$. Let $\tau(p_{01}), \tau(p_{02}) \in \tau(\Upsilon_{(\zeta,\delta)})$. Then $\eta^2(\tau(p_{01})) \geq \zeta$, $\eta^2(\tau(p_{02})) \geq \zeta$, $\hat{\eta^2}(\tau(p_{01})) \leq \delta$ and $\hat{\eta^2}(\tau(p_{02})) \leq \delta$. Then

$$\begin{aligned}\eta^2(\tau(p_{01})\tau(p_{02})^{-1}) &= \eta^2(\tau(p_{01})\tau(p_{02}^{-1})) \\ &= \eta^2(\tau(p_{01}p_{02}^{-1})) \\ &\geq \bar{\Upsilon}^2(p_{01}p_{02}^{-1})\end{aligned}$$

Since $p_{01}, p_{02} \in \Upsilon_{(\zeta,\delta)}$ and $\Upsilon_{(\zeta,\delta)}$ is an HX-SG, then $p_{01}p_{02}^{-1} \in \Upsilon_{(\zeta,\delta)}$ which implies that $\bar{\Upsilon}^2(p_{01}p_{02}^{-1}) \geq \zeta$. Thus $\eta^2(\tau(p_{01})\tau(p_{02})^{-1}) \geq \zeta$. In addition,

$$\begin{aligned}\hat{\eta^2}(\tau(p_{01})\tau(p_{02})^{-1}) &= \hat{\eta^2}(\tau(p_{01})\tau(p_{02}^{-1})) \\ &= \hat{\eta^2}(\tau(p_{01}p_{02}^{-1})) \\ &\leq \hat{\Upsilon}^2(p_{01}p_{02}^{-1})\end{aligned}$$

Since $p_{01}, p_{02} \in \Upsilon_{(\zeta,\delta)}$ and $\Upsilon_{(\zeta,\delta)}$ is an HX-SG, it implies that $\hat{\Upsilon}^2(p_{01}p_{02}^{-1}) \leq \delta$. Thus $\hat{\eta^2}(\tau(p_{01})\tau(p_{02})^{-1}) \leq \delta$. Therefore, $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-SG.

Theorem 14. *Let $\tau : P \longrightarrow S$ be a homomorphism of HX-groups and let Υ^2 be a PFSS of W. If $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-NSG.*

Proof. Since $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then it is an HX-SG and, by the previous theorem, $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-SG. We need to prove that it is normal. Suppose that $\tau(\Upsilon_{(\zeta,\delta)}) = \{(\tau(p), \eta^2, \hat{\eta^2}) : p \in \Upsilon_{(\zeta,\delta)}\}$. Let $\tau(p_{01}), \tau(p_{02}) \in \tau(\Upsilon_{(\zeta,\delta)})$. Then $\eta^2(\tau(p_{01})) \geq \zeta$, $\eta^2(\tau(p_{02})) \geq \zeta$, $\hat{\eta^2}(\tau(p_{01})) \leq \delta$ and $\hat{\eta^2}(\tau(p_{02})) \leq \delta$. Then

$$\begin{aligned}\eta^2(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) &= \eta^2(\tau(p_{01}^{-1})\tau(p_{02})\tau(p_{01})) \\ &= \eta^2(\tau(p_{01}^{-1}p_{02}p_{01})) \\ &\geq \bar{\Upsilon}^2(p_{01}^{-1}p_{02}p_{01})\end{aligned}$$

Since $p_{01}, p_{02} \in \Upsilon_{(\zeta,\delta)}$ and $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then $p_{01}^{-1}p_{02}p_{01} \in \Upsilon_{(\zeta,\delta)}$. Thus $\bar{\Upsilon}^2(p_{01}^{-1}p_{02}p_{01}) \geq \zeta$. Hence $\eta^2(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) \geq \zeta$. Moreover,

$$\begin{aligned}\hat{\eta^2}(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) &= \hat{\eta^2}(\tau(p_{01}^{-1})\tau(p_{02})\tau(p_{01})) \\ &= \hat{\eta^2}(\tau(p_{01}^{-1}p_{02}p_{01})) \\ &\leq \hat{\Upsilon}^2(p_{01}^{-1}p_{02}p_{01})\end{aligned}$$

Since $p_{01}, p_{02} \in \Upsilon_{(\zeta,\delta)}$ and $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then $p_{01}^{-1}p_{02}p_{01} \in \Upsilon_{(\zeta,\delta)}$. Thus $\hat{\Upsilon}^2(p_{01}^{-1}p_{02}p_{01}) \leq \delta$. Hence $\hat{\eta^2}(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) \leq \delta$. Therefore, $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-NSG.

Theorem 15. *Let $\tau : P \longrightarrow S$ be an antihomomorphism of HX-groups and let Υ^2 be a PFSS of W. If $\Upsilon_{(\zeta,\delta)}$ is an HX-SG, then $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-SG.*

Proof. Suppose that $\tau(\Upsilon_{(\zeta,\delta)}) = \{(\tau(p), \eta^2, \hat{\eta^2}) : p \in \Upsilon_{(\zeta,\delta)}\}$. Let $\tau(p_{01}), \tau(p_{02}) \in \tau(\Upsilon_{(\zeta,\delta)})$. Then $\eta^2(\tau(p_{01})) \geq \zeta$, $\eta^2(\tau(p_{02})) \geq \zeta$, $\hat{\eta^2}(\tau(p_{01})) \leq \delta$ and $\hat{\eta^2}(\tau(p_{02})) \leq \delta$. Then

$$\begin{aligned}\eta^2(\tau(p_{01})\tau(p_{02})^{-1}) &= \eta^2(\tau(p_{01})\tau(p_{02}^{-1}))\\ &= \eta^2(\tau(p_{02}^{-1}p_{01}))\\ &\geq \bar{\Upsilon}^2(p_{02}^{-1}p_{01})\end{aligned}$$

Since $p_{01}, p_{02} \in \Upsilon_{(\zeta,\delta)}$ and $\Upsilon_{(\zeta,\delta)}$ is an HX-SG, then $p_{01}^{-1}, p_{02}^{-1} \in \Upsilon_{(\zeta,\delta)}$ and so $p_{02}^{-1}(p_{01}^{-1})^{-1} \in \Upsilon_{(\zeta,\delta)}$. Now, $\bar{\Upsilon}^2(p_{02}^{-1}p_{01}) = \bar{\Upsilon}^2(p_{02}^{-1}(p_{01}^{-1})^{-1}) \geq \zeta$. Thus $\eta^2(\tau(p_{01})\tau(p_{02})^{-1}) \geq \zeta$. In addition,

$$\begin{aligned}\hat{\eta^2}(\tau(p_{01})\tau(p_{02})^{-1}) &= \hat{\eta^2}(\tau(p_{01})\tau(p_{02}^{-1}))\\ &= \hat{\eta^2}(\tau(p_{02}^{-1}p_{01}))\\ &\leq \hat{\Upsilon}^2(p_{02}^{-1}p_{01})\end{aligned}$$

Since $p_{02}^{-1}(p_{01}^{-1})^{-1} \in \Upsilon_{(\zeta,\delta)}$, it implies that $\hat{\Upsilon}^2(p_{02}^{-1}p_{01}) = \hat{\Upsilon}^2(p_{02}^{-1}(p_{01}^{-1})^{-1}) \leq \delta$. Thus $\hat{\eta^2}(\tau(p_{01})\tau(p_{02})^{-1}) \leq \delta$. Therefore, $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-SG.

Theorem 16. *Let $\tau : P \longrightarrow S$ be a homomorphism of HX-groups and let Υ^2 be a PFSS of W. If $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-NSG.*

Proof. Since $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then it is an HX-SG and, by the previous theorem, $\tau(\Upsilon_{(\zeta,\delta)})$ is an HX-SG. We need to prove that it is normal. Suppose that $\tau(\Upsilon_{(\zeta,\delta)}) = \{(\tau(p), \eta^2, \hat{\eta^2}) : p \in \Upsilon_{(\zeta,\delta)}\}$. Let $\tau(p_{01}), \tau(p_{02}) \in \tau(\Upsilon_{(\zeta,\delta)})$. Then $\eta^2(\tau(p_{01})) \geq \zeta$, $\eta^2(\tau(p_{02})) \geq \zeta$, $\hat{\eta^2}(\tau(p_{01})) \leq \delta$ and $\hat{\eta^2}(\tau(p_{02})) \leq \delta$. Then

$$\begin{aligned}\eta^2(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) &= \eta^2(\tau(p_{01}^{-1})\tau(p_{02})\tau(p_{01}))\\ &= \eta^2(\tau(p_{02}p_{01}^{-1})\tau(p_{01}))\\ &= \eta^2(\tau(p_{01}p_{02}p_{01}^{-1}))\\ &\geq \bar{\Upsilon}^2(p_{01}p_{02}p_{01}^{-1}).\end{aligned}$$

Since $p_{01}, p_{02} \in \Upsilon_{(\zeta,\delta)}$ and $\Upsilon_{(\zeta,\delta)}$ is an HX-NSG, then $p_{01}^{-1}, p_{02} \in \Upsilon_{(\zeta,\delta)}$. This implies that $p_{01}p_{02}p_{01}^{-1} = (p_{01}^{-1})^{-1}p_{02}p_{01}^{-1} \in \Upsilon_{(\zeta,\delta)}$. Thus $\bar{\Upsilon}^2(p_{01}p_{02}p_{01}^{-1}) \geq \zeta$. Hence $\eta^2(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) \geq \zeta$.

Moreover,

$$\begin{aligned}\hat{\eta^2}(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) &= \hat{\eta^2}(\tau(p_{01}^{-1})\tau(p_{02})\tau(p_{01}))\\ &= \hat{\eta^2}(\tau(p_{02}p_{01}^{-1})\tau(p_{01}))\\ &= \hat{\eta^2}(\tau(p_{01}p_{02}p_{01}^{-1}))\\ &\leq \hat{\Upsilon}^2(p_{01}p_{02}p_{01}^{-1}).\end{aligned}$$

Since $p_{01}^{-1}, p_{02} \in \Upsilon_{(\zeta,\delta)}$, then $p_{01}p_{02}p_{01}^{-1} = (p_{01}^{-1})^{-1}p_{02}p_{01}^{-1} \in \Upsilon_{(\zeta,\delta)}$. Thus $\hat{\Upsilon}^2(p_{01}p_{02}p_{01}^{-1}) \leq \delta$. Hence $\hat{\eta^2}(\tau(p_{01})^{-1}\tau(p_{02})\tau(p_{01})) \leq \delta$.

5. Conclusion

This article introduced the novel concept of a pythagorean fuzzy HX-subgroup and a normal HX-subgroup. This study aims to establish the groundwork for a novel theory of pythagorean fuzzy HX-subgroups as it is the extension of fuzzy HX groups and intuitionistic fuzzy HX-subgroup. Various chracterisations for pythagorean fuzzy HX-subgroups and pythagorean normal HX-subgroups are proved. Moreover, the notations of pythagorean fuzzy HX-subgroups homomorphisms and antihomomorphisms are initiated, and some related properties regarding the relationship between a pythagorean fuzzy set and its image are investigated. Characterisations of level pythagorean fuzzy HX-subgroups and normal HX-subgroups are presented.

In future work, this study can be expanded to pythagorean fuzzy soft HX-groups and to apply more strategies for handling other hybrid models, such as m-polar soft HX-groups, bipolar soft HX-groups, and neutrosophic soft HX-groups.

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