EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5782
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

An Introduction to Mixed $\mathcal{H}\big(\theta(\mu,\nu)\big)$-Open Sets Generated by Hereditary Classes in Generalized Topological Spaces

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Abstract. In [1], Kim and Min introduced the operation γ_* and $\mathcal{H}(\theta)$-open sets within the context of generalized topological spaces, utilizing a hereditary class $\mathcal{H}$. In this study, we extend these concepts by employing two generalized topologies, μ and ν, along with a hereditary class $\mathcal{H}$. Specifically, we introduce and investigate the mixed operation $\gamma_*(\mu,\nu)$ (denoted briefly as $\gamma_*(\mu,\nu)$) and the mixed $\mathcal{H}(\theta(\mu,\nu))$-open sets (denoted as $\mathcal{H}(\theta(\mu,\nu))$-open sets). We explore the interrelationships between $\gamma_*(\mu,\nu)$, γ_*, and the μ-closure, as well as the connections between $\mathcal{H}(\theta(\mu,\nu))$-open sets, $\theta(\mu,\nu)$-open sets, and μ-open sets. Additionally, we define the concepts of $\mathcal{H}r(\mu,\nu)$-regular open sets and $\mathcal{H}(\mu,\nu)$-regular open sets. Finally, we examine properties and characterizations of $\mathcal{H}(\theta(\mu,\nu))$-open sets in terms of $\mathcal{H}r(\mu,\nu)$-regular open sets and $\mathcal{H}(\mu,\nu)$-regular open sets.

2020 Mathematics Subject Classifications: 54A05, 54C08s

Key Words and Phrases: Hereditary Classes $\mathcal{H}$, mixed operation $\gamma_*(\mu,\nu)$, mixed $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open sets, $\mathcal{H}r(\mu,\nu)$-regular open sets, $\mathcal{H}(\mu,\nu)$-regular

1. Introduction

Á. Császár formulated the idea of generalized topology and generalized open sets in [2], along with the notion of θ-open sets and their properties. For further details, one can refer to [3–5]. In [6], he also introduced the concept of hereditary classes in generalized topological spaces. Specifically, a subset $\mathcal{H} \subseteq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of a non-empty set X) is termed a hereditary class on X if it satisfies the condition that for any $A \subseteq B$ and $B \in \mathcal{H}$, it follows that $A \in \mathcal{H}$. Building on these foundations of generalized topology and hereditary classes, authors in [1] introduced the concepts of $\mathcal{H}(\theta)$-open sets and the operator γ_*.

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5782

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In this study, we extend these ideas by examining two generalized topologies, denoted as μ and ν, within the framework of a hereditary class $\mathcal{H}$. We introduce and investigate the concepts of the mixed operator $\gamma_*(\mu,\nu)$ (denoted briefly as $\gamma_*(\mu,\nu)$) and mixed $\mathcal{H}(\theta(\mu,\nu))$-open sets (referred to as $\mathcal{H}(\theta(\mu,\nu))$-open sets). Our exploration includes a detailed study of their properties and the relationships between $\gamma_*(\mu,\nu)$, the operator γ_*, and the μ-closure. In addition, we examine the interconnections between sets that are $\mathcal{H}(\theta(\mu,\nu))$-open, $\theta(\mu,\nu)$-open, and μ-open. We also present various properties and characterizations of these concepts in terms of $\mathcal{H}r(\mu,\nu)$-regular open sets and $\mathcal{H}(\mu,\nu)$-regular sets.

2. Preliminaries

Let $X \neq \emptyset$ and let $\mathcal{P}(X)$ be its power set. A family $\mu \subseteq \mathcal{P}(X)$ is called a **generalized topology** (GT) on X if:

$$\emptyset \in \mu,$$
$$\bigcup_{i\in I} U_i \in \mu \quad \text{for any collection } \{U_i\}_{i\in I} \subseteq \mu.$$

This concept was first introduced by Á. Császár in [2]. A pair (X,μ) is then referred to as a **generalized topological space** (GTS) on X. The elements of μ are called μ-**open** sets, while their complements are called μ-**closed** sets. The union of all elements of μ is denoted by $\mathcal{M}_\mu$, as stated in [7]. A GTS (X,μ) is is said to be **strong** [8] if $X \in \mu$. For a subset A of a GTS (X,μ), the μ-**closure** of A, denoted $c_\mu(A)$, is defined as the intersection of all μ-closed sets that contain A. The μ-**interior** of A, denoted $i_\mu(A)$, is the union of all μ-open sets that are contained within A (see [2, 7]).

Now, considering a hereditary class $\mathcal{H}$, an operator $()^* : \mathcal{P}(X) \to \mathcal{P}(X)$ was introduced in [3]. Specifically, $c^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined using $()^*$ by $c^*(A) = A \cup A^*$, where $A^* = \{x \in X \mid A \cap M \notin \mathcal{H}, \forall M \in \mu, x \in M\}$. Here, $x \notin A^*$ if and only if there exists $M \in \mu$ such that $x \in M$ and $M \cap A \in \mathcal{H}$.

Recalling definitions and notations from [3], let μ be a GT on X and $\mathcal{P}(X)$ be the power set of X. A collection $\theta \subseteq \mathcal{P}(X)$ is defined as follows: $A \in \theta$ if for each $x \in A$, there exists $M \in \mu$ containing x such that $M \subseteq c_\mu(M) \subseteq A$. The family θ is a GT on X included in μ, and the elements of θ are $\theta(\mu)$-**open** sets, with complements called $\theta(\mu)$-**closed** sets. For $A \subseteq X$, the operation $\gamma_\theta : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined in [3], by $\gamma_\theta(A) = \{x \in X \mid c_\mu(M) \cap A \neq \emptyset, \forall M \in \mu, x \in M\}$.

Kim and Min in [5], extended the study to θ-open using a hereditary class $\mathcal{H}$: a collection $\mathcal{H}(\theta) \subseteq \mathcal{P}(X)$ is defined such that $A \in \mathcal{H}(\theta)$ if for each $x \in A$, there exists $M \in \mu$ containing x with $M \subseteq c^*_\mu(M) \subseteq A$. The family $\mathcal{H}(\theta)$ is a GT on X included in μ, with elements termed $\mathcal{H}(\theta)$-**open** sets and their complements $\mathcal{H}(\theta)$-**closed** sets. Additionally, the operation $\gamma_* : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined in [5] as $\gamma_*(A) = \{x \in X \mid c^*_\mu(M) \cap A \neq \emptyset, \forall M \in \mu, x \in M\}$.

In [4], Á. Császár and Makai Jr. introduced $\theta(\nu_1,\nu_2)$-open sets as a means of combining two generalized topologies (GTs), ν_1 and ν_2, on a set X. A subset $A \subseteq X$ belongs to

$\theta(\nu_1, \nu_2)$ if, for every $x \in A$, there exists $M \in \nu_1$ such that $x \in M \subseteq c_{\nu_2}(M) \subseteq A$. Moreover, the family $\theta(\nu_1, \nu_2)$ itself forms a GT contained within ν_1 on X. The sets in $\theta(\nu_1, \nu_2)$ are called $\theta(\nu_1, \nu_2)$-open sets, while their complements are referred to as $\theta(\nu_1, \nu_2)$-closed sets.

Subsequently, in [9], Abdo Qahis and Awn Alqahtani introduced a modification of this concept, defining the class of $\tilde{\theta}(\nu_1, \nu_2)$-open sets. A subset $A \subseteq X$ is said to be mixed $\tilde{\theta}(\nu_1, \nu_2)$-open (or simply $\tilde{\theta}(\nu_1, \nu_2)$-open) if, for every $x \in A$, there exists $M \in \nu_1$ such that $x \in M$ and $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$.

In conclusion of this section, we review the following important facts due to their significance to the content of our paper.

Theorem 1. *[6] Let μ be a GTS on X and $\mathcal{H}$ a hereditary class on X. Then $A^* \subseteq c^*_\mu(A) \subseteq c_\mu(A)$ for any $A \subseteq X$.*

In [10], the authors introduced the operator $i^* : \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $i^*(A) = X \setminus c^*(X \setminus A)$ for $A \subseteq X$.

Theorem 2. *[10] Let μ be a GT on X and $\mathcal{H}$ a hereditary class. Then for $A \subseteq X$,*

(i) $c^*_\mu(A) = X \setminus i^*_\mu(X \setminus A)$.

(ii) $i_\mu(A) \subseteq i^*_\mu(A) \subseteq A$.

Lemma 1. *[11] Let μ and ν be two GTs on a nonempty set X and $A \subseteq X$. Then the following statements hold:*

(i) $x \in i_{\theta(\mu,\nu)}(A)$ if and only if there exists a μ-open set M containing x such that $M \subseteq c_\nu(M) \subseteq A$.

(ii) If A is ν-open in X, then $\gamma_{\theta(\mu,\nu)}(A) = c_\mu(A)$.

Definition 1. *[1] Let μ be GT on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. Then (X, μ) is $\mathcal{H}$-regular if and only if for every $x \in X$ and every μ-open set U containing x, there exists a μ-open set V containing x such that $x \in V \subseteq c^*(V) \subseteq U$.*

Theorem 3. *[11] Let μ and ν be measures on a nonempty set X. Then X is (μ, ν)-regular if and only if for every $x \in X$ and every μ-open set U containing x, there exists a μ-open set V containing x such that $x \in V \subseteq c_\nu(V) \subseteq U$.*

3. Properties of the Mixed Operator $\gamma_*(\mu, \nu)$

In [4], Császár and Makai Jr introduced an operation $\gamma_{\theta(\mu,\nu)} : \mathcal{P}(X) \to \mathcal{P}(X)$, utilizing two generalized topologies μ and ν on X. According to their definition, $x \in \gamma_{\theta(\mu,\nu)}(A)$ if and only if $c_\nu(M) \cap A \neq \emptyset$ for every μ-open set M containing x. If $x \notin \mathcal{M}_\mu$, then by definition $x \in \gamma_{\theta(\mu,\nu)}(A)$. Additionally, $x \notin \gamma_{\theta(\mu,\nu)}(A)$ if and only if there exists $M \in \mu$ with $x \in M$ such that $c_\nu(M) \cap A = \emptyset$.

Definition 2. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. An operation $\gamma_*(\mu,\nu) : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: for every $A \subseteq X$,*

$$\gamma_*(\mu,\nu)(A) = \{x \in X : c^*_\nu(M) \cap A \neq \emptyset,\ \forall M \in \mu \text{ and } x \in M\}.$$

If $x \notin \mathcal{M}_\mu$, then by definition $x \in \gamma_*(\mu,\nu)(A)$.

According to Definition 2, $x \notin \gamma_*(\mu,\nu)(A)$ if and only if there exists $M \in \mu$ and $x \in M$ such that $c^*_\nu(M) \cap A = \emptyset$.

The following is an immediate consequence that can be obviously obtained.

Corollary 1. *Let μ and ν be two GT's on a nonempty set X such that $\mu = \nu$, and let $\mathcal{H}$ be a hereditary class on X. For any $A \subseteq X$, the following statements hold:*

(i) $\gamma_*(\mu,\nu)(A) = \gamma_*(A)$.

(ii) *If $\mathcal{H} = \{\emptyset\}$, then* $\gamma_*(\mu,\nu)(A) = \gamma_*(A) = \gamma_\theta(A)$.

Theorem 4. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. Then for any $A \subseteq X$, we have $\gamma_*(\mu,\nu)(A) \subseteq \gamma_{\theta(\mu,\nu)}(A)$.*

Proof. Let $x \in \gamma_*(\mu,\nu)(A)$. For each μ-open set M containing x, we have $c^*_\nu(M) \cap A \neq \emptyset$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, it follows that $c_\nu(M) \cap A \neq \emptyset$. Therefore, $x \in \gamma_{\theta(\mu,\nu)}(A)$, and so $\gamma_*(\mu,\nu)(A) \subseteq \gamma_{\theta(\mu,\nu)}(A)$.

The following example demonstrates that, in general, $\gamma_*(\mu,\nu)(A) \neq \gamma_\theta(\mu,\nu)(A)$.

Example 1. *Let $X = \{a,b,c,d\}$. Consider two generalized topologies: $\mu = \{\emptyset, \{b,d\}\}$ and $\nu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}\}$ on X.*

For a set $A = \{a,c\}$, we have $c_\nu(\{b,d\}) = X$, $\mathcal{M}_\mu = \{b,d\}$, and $c_\nu(\{b,d\}) \cap A \neq \emptyset$. Thus, $\gamma_{\theta(\mu,\nu)}(A) = X$. Since $\mathcal{M}_\mu = \{b,d\}$, it is clear that $a, c \in \gamma_*(\mu,\nu)(A)$. Noting that $\{b,d\}^*(\mathcal{H},\nu) = \{d\}$, we find $c^*_\nu(\{b,d\}) \cap A = \emptyset$, hence $b, d \notin \gamma_*(\mu,\nu)(A)$. Therefore, $\gamma_*(\mu,\nu)(A) = \{a,c\}$ and $\gamma_*(\mu,\nu)(A) \subset \gamma_{\theta(\mu,\nu)}(A)$.

Corollary 2. *Let μ and ν be two GT's on a nonempty set X and let $\mathcal{H}$ be a hereditary class on X. If $\mathcal{H} = \{\emptyset\}$, then $\gamma_*(\mu,\nu)(A) = \gamma_{\theta(\mu,\nu)}(A)$ for any $A \subseteq X$.*

Theorem 5. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. For any subsets A and B of X, the following properties hold:*

(i) $\gamma_*(\mu,\nu)(\emptyset) = \emptyset$.

(ii) *If $A \subseteq B$, then* $\gamma_*(\mu,\nu)(A) \subseteq \gamma_*(\mu,\nu)(B)$.

(iii) $A \subseteq c_\mu(A) \subseteq \gamma_*(\mu,\nu)(A)$.

Proof. (1) and (2) are obvious.

(3) For $x \in c_\mu(A)$ and any μ-open set M containing x, we have $M \cap A \neq \emptyset$. Consequently, $c^*_\nu(M) \cap A \neq \emptyset$. Therefore, $x \in \gamma_*(\mu,\nu)(A)$, implying that $c_\mu(A) \subseteq \gamma_*(\mu,\nu)(A)$.

The following example shows that, in general, $c_\mu(A) \neq \gamma_*(\mu,\nu)(A)$.

Example 2. *Let $X = \{a, b, c, d\}$. Consider two generalized topologies: $\mu = \{\emptyset, \{a\}\}$, $\nu = \big\{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\big\}$ on X, and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}\}$.*

For a set $A = \{b, c, d\}$, since A is μ-closed, $c_\mu(A) = A$. Given $\mathcal{M}_\mu = \{a\}$, it follows by the definition of the operator $\gamma_*(\mu, \nu)$ that $X - \mathcal{M}_\mu = \{b, c, d\} \subseteq \gamma_*(\mu, \nu)(A)$.

Next, we show that $a \in \gamma_*(\mu, \nu)(A)$. Since $M = \{a\} \in \mu$ and $\{a\}^*(\mathcal{H}, \nu) = \{a, d\}$, we have $c^*_\nu(M) \cap A \neq \emptyset$. Thus, $a \in \gamma_*(\mu, \nu)(A)$, implying $c_\mu(A) \subset \gamma_*(\mu, \nu)(A) = X$. Thus $c_\mu(A) \neq \gamma_*(\mu, \nu)(A)$.

The following Corollary follows immediately from Theorem 5(iii), and Theorem 1.

Corollary 3. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. For $A \subseteq X$, $A^* \subseteq c^*_\mu(A) \subseteq \gamma_*(\mu, \nu)(A)$.*

Theorem 6. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. Then $\gamma_*(\mu, \nu)(A)$ is μ-closed.*

Proof. Let $x \in X - \gamma_*(\mu, \nu)(A)$. This means there exists $M_x \in \mu$ such that $c^*_\nu(M_x) \cap A = \emptyset$. Since $M_x \subseteq c^*_\nu(M_x)$, it follows that $M_x \cap A = \emptyset$. Therefore, every $y \in M_x$ implies $y \in X - \gamma_*(\mu, \nu)(A)$, implying $X - \gamma_*(\mu, \nu)(A) = \bigcup_{x \in X - \gamma_*(\mu,\nu)(A)} M_x$. Thus, $X - \gamma_*(\mu, \nu)(A)$ is μ-open, hence $\gamma_*(\mu, \nu)(A)$ is μ-closed.

Theorem 7. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. If A is ν-open in X, then $\gamma_*(\mu, \nu)(A) = c_\mu(A)$.*

Proof. From (iii) of Theorem 5, we have $c_\mu(A) \subseteq \gamma_*(\mu, \nu)(A)$.

For the converse inclusion, suppose $x \in \gamma_*(\mu, \nu)(A)$. For each $M \in \mu$ such that $x \in M$ and $c^*_\nu(M) \cap A \neq \emptyset$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, it follows that $c_\nu(M) \cap A \neq \emptyset$. Thus, there exists $y \in c_\nu(M) \cap A$. Since A is ν-open and contains y, we have $M \cap A \neq \emptyset$, implying $x \in c_\mu(A)$. Therefore, $\gamma_*(\mu, \nu)(A) \subseteq c_\mu(A)$. Combining this with the earlier inclusion, we conclude $\gamma_*(\mu, \nu)(A) = c_\mu(A)$.

The following Corollary follows from Lemma 1(ii) and Theorem 7.

Corollary 4. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. If $A \in \nu$, then*

$$\gamma_*(\mu, \nu)(A) = c_\mu(A) = \gamma_{\theta(\mu,\nu)}(A).$$

4. $\mathcal{H}\big(\theta(\mu, \nu)\big)$-Open Sets

Definition 3. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. We define the collection $\mathcal{H}\big(\theta(\mu, \nu)\big) \subseteq \mathcal{P}(X)$ such that $A \in \mathcal{H}\big(\theta(\mu, \nu)\big)$ if and only if for each $x \in A$, there exists $M \in \mu$ such that $x \in M \subseteq c^*_\nu(M) \subseteq A$.*

The elements of $\mathcal{H}\big(\theta(\mu, \nu)\big)$ are called mixed $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open (briefly, $\mathcal{H}\big(\theta(\mu, \nu)\big)$-open) , and their complements are called mixed $\mathcal{H}\big(\theta(\mu, \nu)\big)$-closed (briefly, $\mathcal{H}\big(\theta(\mu, \nu)\big)$-closed).

Remark 1. *Consider μ and ν to be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. If $\mu = \nu$, then $\mathcal{H}\big(\theta(\mu,\nu)\big) = \mathcal{H}(\theta)$.*

Theorem 8. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. Then $\theta(\mu,\nu) \subseteq \mathcal{H}\big(\theta(\mu,\nu)\big) \subseteq \mu$.*

Proof. To show that $\theta(\mu,\nu) \subseteq \mathcal{H}\big(\theta(\mu,\nu)\big)$, let $A \in \theta(\mu,\nu)$ and $x \in A$. Then there exists $M \in \mu$ such that $x \in M \subseteq c_\nu(M) \subseteq A$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, we have $x \in M \subseteq c^*_\nu(M) \subseteq A$. Therefore, A is an $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open set.

Next, to show that $\mathcal{H}\big(\theta(\mu,\nu)\big) \subseteq \mu$, suppose $A \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ and let $x \in A$. Then there exists a μ-open set M_x such that $x \in M_x \subseteq c^*_\nu(M_x) \subseteq A$. Therefore, $A = \bigcup_{x\in A} M_x \in \mu$.

Remark 2. *Based on Theorem 8, we can illustrate the following diagram.*

$$\begin{array}{ccc} \mathcal{K}(\theta(\mu,\nu)) & \longrightarrow & \mu\text{-open} \\ \uparrow & & \downarrow \\ \theta(\mu,\nu) & \longrightarrow & \mu^*\text{-open} \end{array}$$

The following example demonstrates that the above implications are not reversible in general.

Example 3. *Let $X = \{a,b,c,d\}$. Consider two generalized topologies $\mu = \big\{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\big\}$ and $\nu = \big\{\emptyset, \{b,d\}\big\}$ and a hereditary class $\mathcal{H} = \big\{\emptyset, \{b\}\big\}$. Note that:*

(i) For a set $A = \{a,b,c\}$, we have $M_a = M_b = \{a,b\} \in \mu$ and $M_c = \{b,c\} \in \mu$. Then $M^*_a(\mathcal{H},\nu) = M^*_b(\mathcal{H},\nu) = M^\star_c(\mathcal{H},\nu) = \{a,c\}$ and $c^*_v(M_a) = c^*_v(M_b) = c^*_v(M_c) = \{a,b,c\} \subseteq A$;

(ii) Since $c_\nu(\{a,b\}) = c_\nu(\{b,c\}) = c_\nu(\{a,b,c\}) = X \nsubseteq A$, then $\theta(\mu,\nu) = \{\emptyset\}$.

(iii) From (1) and (2), we show that the set A is $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open but it is not $\theta(\mu,\nu)$-open. Also, it is easy to check that $B = \{a,b\}$ is μ-open but it is not $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open.

Theorem 9. *Let μ and ν be two GTs on a nonempty set X, and $\mathcal{H}$ be a hereditary class on X. Then the family $\mathcal{H}\big(\theta(\mu,\nu)\big)$ is a GT contained in μ on X.*

Proof. Firstly, $\emptyset \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ is obvious. Now, let $\{A_\alpha \subseteq X : A_\alpha \in \mathcal{H}\big(\theta(\mu,\nu)\big)\}$ for $\alpha \in \Lambda$. Consider $x \in \cup_\alpha A_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that for some μ-open set M containing x, we have $M \subseteq c^*_\nu(M) \subseteq A_{\alpha_0}$. This implies there exists $x \in M \in \mu$ such that $M \subseteq c^*_\nu(M) \subseteq \cup_\alpha A_\alpha$ and so $\cup_\alpha A_\alpha \in \mathcal{H}\big(\theta(\mu,\nu)\big)$.

Theorem 10. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. Then, A is $\mathcal{H}(\theta(\mu,\nu))$-closed if and only if $\gamma_*(\mu,\nu)(A) = A$.*

Proof. Let A be $\mathcal{H}\big(\theta(\mu,\nu)\big)$-closed in X. Since $X-A \in \mathcal{H}\big(\theta(\mu,\nu)\big)$, for each $x \in X-A$, there exists $M \in \mu$ such that $x \in M \subseteq c_\nu^*(M) \subseteq X-A$. Thus, $c_\nu^*(M) \cap A = \emptyset$, implying $x \notin \gamma_*(\mu,\nu)(A)$. Therefore, $\gamma_*(\mu,\nu)(A) \subseteq A$, implying that $\gamma_*(\mu,\nu)(A) = A$.

For the reverse inclusion, suppose $\gamma_*(\mu,\nu)(A) = A$ and let $x \in X-A = X-\gamma_*(\mu,\nu)(A)$. Then there exists $M \in \mu$ such that $x \in M$ and $c_\nu^*(M) \cap A = \emptyset$. Hence, $x \in M \subseteq c_\nu^*(M) \subseteq X-A$, showing that $X-A$ is $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open. Therefore, A is $\mathcal{H}\big(\theta(\mu,\nu)\big)$-closed.

From Theorem 10 and Theorem 7, the following Corollary is directly obtained.

Corollary 5. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$ be $\mathcal{H}\big(\theta(\mu,\nu)\big)$-closed. If $A \in \nu$, then A is μ-closed.*

Definition 4. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. The $\mathcal{H}\big(\theta(\mu,\nu)\big)$-closure of $A \subseteq X$, denoted by $c_{\mathcal{H}\theta(\mu,\nu)}(A)$, is the intersection of all $\mathcal{H}\big(\theta(\mu,\nu)\big)$-closed sets containing A. The $\mathcal{H}\big(\theta(\mu,\nu)\big)$-interior of A, denoted by $i_{\mathcal{H}(\theta(\mu,\nu))}(A)$, is the union of all $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open sets contained in A.*

Theorem 11. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. Then $\gamma_*(\mu,\nu)(A) \subseteq c_{\mathcal{H}\theta(\mu,\nu)}(A)$.*

Proof. Let $x \notin c_{\mathcal{H}\theta(\mu,\nu)}(A)$. Then there exists an $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open set W containing x such that $W \cap A = \emptyset$. Since $W \in \mathcal{H}\big(\theta(\mu,\nu)\big)$, there exists $M \in \mu$ containing x such that $x \in M \subseteq c_\nu^*(M) \subseteq W \subseteq X-A$. This implies $c_\nu^*(M) \cap A = \emptyset$ and hence $x \notin \gamma_*(\mu,\nu)(A)$. Therefore, $\gamma_*(\mu,\nu)(A) \subseteq c_{\mathcal{H}\theta(\mu,\nu)}(A)$.

Definition 5. *Let μ and ν be two GTs on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. A subset $A \subseteq X$ is called $\mathcal{H}(\mu,\nu)$-regular open (briefly, $\mathcal{H}r(\mu,\nu)$-open) if $A = i_\mu(c_\nu^*(A))$. Similarly, A is called $\mathcal{H}(\mu,\nu)$-regular closed (briefly, $\mathcal{H}r(\mu,\nu)$-closed) if $c_\mu(i_\nu^*(A)) = A$.*

Theorem 12. *Let μ and ν be two GTs on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. If $A \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ and $x \in A$, then there exists a $\mathcal{H}r(\mu,\nu)$-open set U such that $x \in U \subseteq c_\nu^*(U) \subseteq A$.*

Proof. Since $A \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ and $x \in A$, there exists a μ-open set M such that $x \in M \subseteq c_\nu^*(M) \subseteq A$. Define $U = i_\mu\big(c_\nu^*(M)\big)$. Then U is $\mathcal{H}r(\mu,\nu)$-open, $M \subseteq U$, and $c_\nu^*(U) = c_\nu^*\big(i_\mu\big(c_\nu^*(M)\big)\big) \subseteq c_\nu^*(M)$. This implies $x \in M \subseteq U \subseteq c_\nu^*(U) \subseteq c_\nu^*(M) \subseteq A$. Thus, we have $x \in U \subseteq c_\nu^*(U) \subseteq A$ for some $\mathcal{H}r(\mu,\nu)$-open set U.

Since every $\mathcal{H}(\mu,\nu)$-regular open set is μ-open in X, the following Corollary is evidently obtained.

Corollary 6. *Let μ and ν be two GT's on a nonempty set X, $\mathcal{H}$ a hereditary class on X, and $A \subseteq X$. Then, $A \in \mathcal{H}\big(\theta(\mu,\nu)\big)$ and $x \in A$ if and only if there exists a $\mathcal{H}r(\mu,\nu)$-open set U such that $x \in U \subseteq c_\nu^*(U) \subseteq A$.*

Definition 6. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. A set X is said to be $\mathcal{H}(\mu,\nu)$-regular (or simply $\mathcal{H}(\mu,\nu)$-regular) if for every $x \in X$ and every μ-closed set F with $x \notin F$, there exist sets $U \in \mu$, $V \in \nu^*$ such that:*

$$x \in U, \quad F \subseteq V, \quad \textit{and} \quad U \cap V = \emptyset.$$

Theorem 13. *Let μ and ν be GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. Then X is $\mathcal{H}(\mu,\nu)$-regular if and only if for every $x \in X$ and every μ-open set U containing x, there exists a μ-open set V containing x such that $x \in V \subseteq c^*_\nu(V) \subseteq U$.*

Proof. Assume X is $\mathcal{H}(\mu,\nu)$-regular. For $x \in X$ and a μ-open set U containing x, there exist disjoint sets $V \in \mu$ and $W \in \nu^*$ such that $x \in V$, $(X-U) \subseteq W$. Since $V \subseteq X-W$ and $X-W$ is ν^*-closed, we have $c^*_\nu(V) \subseteq X-W$. This implies $c^*_\nu(V) \cap (X-U) \subseteq c^*_\nu(V) \cap W = \emptyset$, hence $x \in V \subseteq c^*_\nu(V) \subseteq U$.

Conversely, suppose F is a μ-closed set and $x \notin F$ for $x \in X$. Since $X - F$ is a μ-open set containing x, by hypothesis, there exists a μ-open set V containing x such that $x \in V$, $V \subseteq c^*_\nu(V) \subseteq X - F$, $c^*_\nu(V) \cap F = \emptyset$, and $F \subseteq X - c^*_\nu(V)$. As $X - c^*_\nu(V) \in \nu^*$ and $V \cap \big(X - c^*_\nu(V)\big) = \emptyset$, it follows that X is $\mathcal{H}(\mu,\nu)$-regular.

Remark 3. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X such that $\mu = \nu$. If X is $\mathcal{H}(\mu,\nu)$-regular, then X is also $\mathcal{H}$-regular.*

Proposition 1. *Let μ and ν be two GT's on a nonempty set X, and $\mathcal{H}$ a hereditary class on X. If X is (μ,ν)-regular, then X is $\mathcal{H}(\mu,\nu)$-regular.*

Proof. Let X be (μ,ν)-regular. Consider $x \in X$ and an μ-closed set F such that $x \notin F$. Then $X - F$ is a μ-open set containing x. By Theorem 3, there exists a μ-open set V containing x such that

$$x \in V \subseteq c_\nu(V) \subseteq X - F.$$

Since $c^*_\nu(V) \subseteq c_\nu(V)$, it follows that

$$x \in V \subseteq c^*_\nu(V) \subseteq X - F.$$

By Theorem 13, X is $\mathcal{H}(\mu,\nu)$-regular.

Theorem 14. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. If X is $\mathcal{H}(\mu,\nu)$-regular, then the following hold:*

(i) For any $A \subseteq X$, $\gamma_*(\mu,\nu)(A) = c_\mu(A)$.

(ii) Every μ-open set is $\mathcal{H}(\theta(\mu,\nu))$-open.

Proof. (1) By Theorem 5(iii), we have $c_\mu(A) \subseteq \gamma_*(\mu,\nu)(A)$. To show the reverse inclusion, let $x \in \gamma_*(\mu,\nu)(A)$ and let U be any μ-open set containing x. From $\mathcal{H}(\mu,\nu)$-regularity, there exists a μ-open set V such that $x \in V \subseteq c^*_\nu(V) \subseteq U$. Since $x \in \gamma_*(\mu,\nu)(A)$, it follows that $c^*_\nu(V) \cap A \neq \emptyset$. Thus, $U \cap A \neq \emptyset$, implying $x \in c_\mu(A)$.

(2) Let M be a μ-open set. From (1), we have $\gamma_*(\mu,\nu)(X-M) = c_\mu(X-M) = X-M$. By Theorem 10, $X - M$ is $\mathcal{H}(\theta(\mu,\nu))$-closed, which means M is $\mathcal{H}(\theta(\mu,\nu))$-open.

The next result follows from Theorem 8 and Theorem 14(ii).

Corollary 7. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. If X is $\mathcal{H}(\mu,\nu)$-regular, then $\mu = \mathcal{H}(\theta(\mu,\nu))$.*

Definition 7. *Let μ and ν be two GT's on a nonempty set X, and let $\mathcal{H}$ be a hereditary class on X. We define the following notions:*

$$\ell_{\mathcal{H}(\theta(\mu,\nu))(A)} = \{x \in X : c^*_\nu(M) \subseteq A \textit{ for some } \mu\textit{-open set } M \textit{ containing } x\}.$$

$$\ell_{\theta(\mu,\nu))(A)} = \{x \in X : c_\nu(M) \subseteq A \textit{ for some } \mu\textit{-open set } M \textit{ containing } x\}.$$

$$\ell_{\mathcal{H}(\theta)(A)} = \{x \in X : c^*_\mu(M) \subseteq A \textit{ for some } \mu\textit{-open set } M \textit{ containing } x\}.$$

Proposition 2. *For any two GT's ν_1 and ν_2 on a nonempty set X, we have $\ell_{\theta(\nu_1,\nu_2)}(A) \subseteq \ell_{\mathcal{H}(\theta(\mu,\nu))(A)}$ for any $A \subseteq X$.*

Proof. Let $x \in \ell_{\theta(\mu,\nu)}(A)$. Then there exists a μ-open set M containing x such that $c_\nu(M) \subseteq A$. Since $c^*_\nu(M) \subseteq c_\nu(M)$, it follows that $c^*_\nu(M) \subseteq A$. Therefore, $x \in \ell_{\mathcal{H}(\theta(\mu,\nu))(A)}$.

Remark 4. *Let μ and ν be two GTs on a nonempty set X, and let $A \subseteq X$. If $\mu = \nu$, then $\ell_{\mathcal{H}(\theta(\mu,\nu))(A)} = \ell_{\mathcal{H}(\theta)(A)}$.*

Theorem 15. *Let ν_1 and ν_2 be two GT's on a nonempty set X and $A \subseteq X$. Then the following properties hold:*

(i) $i_{\mathcal{H}(\theta(\mu,\nu)}(A) = X - c_{\mathcal{H}(\theta(\mu,\nu)}(X - A)$ *and* $c_{\mathcal{H}(\theta(\mu,\nu)}(A) = X - i_{\mathcal{H}(\theta(\mu,\nu)}(X - A)$.

(ii) $\ell_{\mathcal{H}(\theta(\mu,\nu)}(A) = X - \gamma_*(\mu,\nu)(X - A)$ *and* $\gamma_*(\mu,\nu)(A) = X - \ell_{\mathcal{H}(\theta(\mu,\nu)}(X - A)$.

Proof. The proof is obvious.

The following Corollary comes directly from Definition 4 and Definition 7.

Corollary 8. *Let μ and ν be two GTs on a nonempty set X and $A \subseteq X$. Then $i_{\mathcal{H}(\theta(\mu,\nu))}(A)$ if and only if there exists a μ-open set M containing x such that $M \subseteq c^*_\nu(M) \subseteq A$.*

5. Conclusion

This study aimed to introduce and examine the operation $\gamma_*(\mu,\nu)$ and $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open sets within generalized topological spaces. Several significant results regarding these concepts were established. We thoroughly investigated the relationships among $\gamma_*(\mu,\nu)$, γ_*, and the μ-closure, as well as those among $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open sets, $\theta(\mu,\nu)$-open sets, and μ-open sets. Finally, we have derived various properties and characterizations of $\mathcal{H}\big(\theta(\mu,\nu)\big)$-open sets in terms of the concept of $\mathcal{H}(\mu,\nu)$-regularity.

Acknowledgements

We would like to thank the reviewers for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments, careful reading and suggestions that lead to improve the quality of this manuscript.

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