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# A Breakthrough Approach to Quadri-Partitioned Neutrosophic Soft Topological Spaces

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**Abstract.** Neutrosophic set theory, an advanced framework for error reduction, extends fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs). It enhances effectiveness by refining the definition of indeterminacy, a concept for situations where values cannot be precisely determined. In this paper, we propose dividing indeterminacy into two components based on membership: relative truth (RT), which leans toward truth, and relative falsehood (RF), which leans toward falsehood. This approach improves accuracy by considering, relative truth and relative falsehood, rather than a single indeterminate value. The modified neutrosophic set, called the quadri-portioned neutrosophic soft set (QPNSS), includes four membership attributes: absolute truth, relative truth, relative falsehood, and absolute falsehood, offering greater clarity in uncertain situations. New operations are introduced on QPNSS, such as the quadri-partitioned neutrosophic soft set, subsets, complement, absolute set, set difference, and null set. AND and OR operations are also defined. Additionally, a quadri-partitioned neutrosophic soft topological space (QPNSTS) is defined, with key results presented. We define and explore new concepts such as pre-open (p-open) sets, interior, and closure. The paper also examines QPNS compactness, reducibility to finite sub-covers, and other important properties like the intersection of QPNS p-closed sets and QPNS p-compact spaces. This work contributes to the theoretical understanding of QPNS spaces, particularly in soft topological spaces.

**2020 Mathematics Subject Classifications**: 54A05

**Key Words and Phrases**: Neutrosophic soft set, QPNSS, QPNSTS, QPNS p- open sets, QPNS p-closed sets, QPNS p compact subsets.

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## 1. Introduction

The most significant method for dealing with ambiguity and missing data that arise in many scientific domains is fuzzy set theory (FST) [1]. This theory is only helpful when discussing membership value; it cannot sufficiently address the non-membership value. This challenge was expertly resolved by Atanassov [2], who introduced the concept of intuitionistic fuzzy set theory (IFST). This method captures the value of membership as well as non-membership. The concept of soft set theory (SST) was first proposed by Molodtsov [3] as a novel way to handle nebulous and confusing circumstances. Molodtsov [4] tackled a number of problems, including function smoothness, using soft set theory. Xu et al. [5] initially presented vague soft set theory as an expansion of FST. In their discussion of unclear soft set, Huang et al. [6] identified a few inaccurate findings. The authors provided additional new definitions and provided examples to support the wrong finding. The idea of vague soft topological space was first introduced by Chang Wang and Yaya Li [7] under the heading topological structure of vague soft sets. The authors examined the findings in this area and talked about the fundamental ideas of vague soft topology. Smarandache [8] extended soft set to hyper soft set. The author also presented the hybrids of NHSSs, IFSSs, PHSSs, and crisp FSs. Based on the neutrosophic soft set (NSS). Bera and Mahapatra [9] proposed the idea of a new structure known as neutrosophic soft topology. This concept opened up a whole new area of mathematics. The writers courteously went over each of the foundational concepts before moving on to the fundamental outcomes and providing pertinent instances for greater comprehension. Ozturk et al. [10] installed extremely operations on NSSs and NSTs based on these unique procedures. Based on the operation outlined in [10], Ozturk et al. [11] installed the notions of neutrosophic soft mapping, neutrosophic soft open mapping, and neutrosophic soft homeomorphism. Mehmood et al. [12] created new open sets in neutrosophic soft topological space. AL-Nafee [13] not only created a new family of neutrosophic soft sub sets but also came up with new operations (union and intersection) on neutrosophic sets. Based on the operations described in [14], AL-Nafee et al. [14] constructed NSBTS to finish their previous research. The authors used these fundamental methods to replicate all of the important findings of NSBTS. NSBTS was developed by Dadas and Demiralp [15] using the guidelines provided in [9]. Smarandache [16] generalizes the intuitionistic fuzzy set (IFS), paraconsistent set, and intuitionistic set to the neutrosophic set (NS). Many examples are presented. Distinctions between NS and IFS are underlined. Several aspects of the menger selection principle were explored by Ljubia et al. [17] in their study of the principle in regard to soft sets. The relationships between the recently suggested soft open coverings were examined by the writers. Using soft s-regular open covers, Al-shami et al. [18] provided a detailed description of near SMSs. Additionally; they demonstrated how they fell into the same class as soft regular spaces. Al-shami et al.'s concept of an ISTSs was first presented [19]. Al-shami et al. [20] explored fixed soft points and weak Forms of soft separation axioms. The theory of ISS connected and ISLCS was given by Al-shami et al. [21]. They also discussed the behavior of these spaces as a finite product of soft spaces and under infra soft homeomorphism maps. The authors discussed the essential elements of a soft point and described its main traits. The concepts of open, closed, and homeomorphism mappings in ISTS content were first introduced by Al-shami [22].

Al-shami [23] investigated ISC and ISLSs, demonstrating their main features using a sequence of infra soft closed sets. Examining the transmission of these concepts between classical and IST was the author's main focus. The concept of soft bi-operators was first introduced by Baravan et al. [24]. An example from Ali et al. was used to build an effective bipolar soft generalization of the q-rung ortho-pair fuzzy set model [25]. A model known as the FBSESs was introduced by Ali et al. [26]. In order to create a unique discretization of these basic mathematical ideas with an essentially continuous character, John [27] proposed a soft structure over a set. The advantage is that it gave rise to new instruments for applying mathematical analysis technology in practical applications to identify uncertainty or incomplete data. S. Kim and Park [28] discussed secure multi-party clustering Protocols. Kandasamy [29] installed DVNSs and used it minimum spanning trees. Mehmood et al. [30] explored many structures and provided new definitions for NSTSs. Mehmood et al. [31] presented the idea of VSTSs in a novel approach and explored various structures with regard to vague soft points using vague soft beta open sets.

## 1.1. Literature review

A novel idea known as vague soft bi-topological space was presented by Mehmood et al. [32], who also looked at its structural characteristics. Generalized vague soft open sets, also known as vague soft $\beta$-open sets, are the foundation of this method. These structures are demonstrated with a number of instances. The authors used vague soft $\beta$-open sets to study the properties of vague soft bi-topological space with respect to the soft points of the spaces. The Bolzano-Weierstrass property, vague soft compactness and its link to sequences, the characterization of vague soft $\beta$-closed and vague soft $\beta$-opensets, and the relationship between countable compactness and sequential compactness in VSBTS with respect to soft $\beta - open$ sets are also discussed. New union, intersection, and complement operations were introduced by Mehmood et al. [33] and serve as the basis for the definition of vague soft bi-topological space. Within vague soft bi-topological space, they created pairwise vague soft closed sets and pairwise vague soft open sets. They also presented generalized vague soft open sets associated with the space's soft points. They established separation axioms based on these generalized ambiguous soft open sets. Furthermore, other noteworthy findings in vague soft bi-topological space are connected to these separation axioms. The notions of neutrosophic soft b-closed sets and neutrosophic soft b-open sets, as well as their attributes, were first presented by Mehmood et al. [34]. In connection with soft points, they also talked about neutrosophic soft b-neighborhoods and neutrosophic soft b-separation axioms in NSTS. Numerous findings pertaining to soft points are used to investigate the idea of neutrosophic soft b-separation axioms. Furthermore, the characteristics and relationships of neutrosophic soft $\mathrm{T}_i$ spaces for ($i$ = 0,1,2,3,4) are investigated. Mehmood et al. [35] discussed NSQS concerning soft points. The novel idea of QPNSS was presented by Ramesh and Mary [36], who also talked about some of its characteristics. For clarity and comprehension, the writers picked the best instances and concentrated on a methodical investigation. Shami and Mhemdi [19] explored two families of separation axioms in the context of infra soft topological spaces. Their work aimed at classifying and analyzing different types of separation properties within this specialized topological framework. Shami and Liu [19] introduced two classes of infra-soft separation axioms. These axioms are important for defining and studying the separation properties of infra-soft topological spaces, extending classical separation axioms to a more generalized soft-topological setting. Shami and Azzam [37] focused on the concepts of soft semi-open sets and infra soft semi continuity. Their study examined how these notions interact with infra soft topological spaces, offering insights into the continuity properties of such spaces. Ahmad et al. [38] discussed Irreversible k-Threshold conversion number for some graph products and neutrosophic graphs. Hatamleh et al. [39] studied complex tangent trigonometric approach applied to $(\Xi, \tau)$-rung fuzzy set using weighted averaging, geometric operators and its extension. Hatamleh et al. [40] studied different weighted operators such as generalized averaging and generalized geometric based on trigonometric $\wp$-rung interval-valued approach and in addition to this some examples were given for clear understanding. Shihadeh et al. [41] discussed algebraic structures towards different $(\alpha, \beta)$ intuitionistic fuzzy ideals and it characterization of an ordered ternary semigroups. Hatamleh et al. [40, 42] studied operators via weighted averaging and geometric approach using trigonometric neutrosophic interval valued set and its extension and characterization of interaction aggregating operators setting interval valued Pythagorean neutrosophic set. Hatamleh et al. [38, 43] discussed applications of complex interval valued picture fuzzy soft relations. . El-Sheikh and Abd El-latif [44] discussed decompositions of some types of supra soft sets and soft continuity and cited some excellent examples for clear understating the concept. Abd El-latif [45] discussed soft supra compactness in supra soft topological spaces. Abd El-latif and Hosny, [46] discussed the eye catching concept of soft separation axioms and give examples. Abd El-latif and Hosny discussed some more structures in [45, 47].

### 1.2. Research Gap

In Neutrosophic set theory, three membership functions truth, indeterminacy, and falsity are commonly used. However, the indeterminacy function presents a significant challenge, as it introduces ambiguity, making it difficult to decisively classify elements. This indeterminate region often leads to errors and confusion in the final results, limiting the applicability and precision of neutrosophic sets in practical scenarios. While some attempts have been made to address this issue, there is still a gap in fully resolving the indeterminacy. A potential solution lies in further subdividing the indeterminacy component into two distinct parts: one that tends toward truth and the other that tends toward falsity. This would effectively eliminate the ambiguity associated with indeterminacy. However, to date, no comprehensive framework has been proposed that incorporates this subdivision. There is a need for the development of a new set structure, which would consist of four components. This could lead to the creation of a quadripartitioned neutrosophic set, and subsequently, the introduction of a novel structure known as the quadripartitioned neutrosophic topological space. This new structure could provide a more precise and error-minimized approach, enabling more accurate operations, properties, and results to be derived from neutrosophic sets. Thus, there exists a clear research gap in formalizing and exploring the properties and applications of quadripartitioned neutrosophic topological spaces.

### 1.3. Motivation

A modified version of a neutrosophic set, called the double-valued neutrosophic Set (DVNS), which consists of two different indeterminate values, was introduced by Kandasamy [31]. Examples are provided to define and demonstrate the related attributes and axioms. In several research domains, such as data mining, pattern recognition, and machine learning, clustering is essential. The cornerstone for building a clustering method is the establishment of a generalized distance measure between DVNSs and the accompanying distance matrix. In order to efficiently cluster data represented by double-valued neutrosophic information, this work suggests the double-valued neutrosophic minimum spanning Tree (DVN-MST) clustering technique. To demonstrate the usefulness and uses of this clustering approach, illustrative examples are given. This work inspired me and served as a stepping stone for my current research.

### 1.4. Novelty

The quadri-portioned neutrosophic soft set (QPNSS) is a novel framework that represents a major breakthrough in neutrosophic set theory. This approach distinguishes between relative truth (RT) and relative falsehood (RF), two essential elements of indeterminacy, improving sensitivity and accuracy in uncertain situations. By enabling a more nuanced examination of values that are difficult to categorize as true or untrue, this innovative classification enhances decision-making. Absolute truth, relative truth, relative falsehood, and absolute falsehood are the four membership traits that make up the QPNSS. Along with quadri-partitioned soft sets and their complements, the study defines new operations on QPNSS, such as AND and OR operations. Moreover, eight new definitions, including the semi-open (s-open) set, and the development of quadri-partitioned neutrosophic soft topological spaces (QPNSTS) improve our knowledge of topological features. Crucially, the research investigates compactness in QPNSS, introducing ideas such as reducibility to finite sub-covers and establishing important theorems on intersection qualities and compact subset separation. These results contribute to the theoretical foundation of QPNS spaces and provide important information about the compactness of soft topological spaces. All things considered, this work opens the door for further study and applications by expanding our knowledge of neutrosophic set theory and offering useful instruments for handling ambiguity. The organization of this paper is structured as follows. We outline basic ideas and findings in Section 2. A new method for operations on quadri-partitioned neutrosophic soft sets is examined in Section 3. A new viewpoint on heptapartitioned neutrosophic soft topological spaces is presented in Section 4. Results on compactness in quadri-partitioned neutrosophic soft topological spaces are presented in Section 5. Comparative analysis is discussed in Section 6. Applications are given in Section 7. Conclusion and future work is given in Section 8.

## 2. Preliminaries

This section covers foundational ideas that are required for the upcoming research. Firstly, neutrosophic soft set defined by Maji [48] and later this concept has been modified by Deli and Bromi [49] as given below:

**Definition 1.** *Let $e$ be a set of parameters and $\mathcal{M}$ be an initial universe set. $P(\mathcal{M})$ represents the collection of all neutrosophic soft sets for $\mathcal{M}$. A set defined by a set-valued function $\tilde{\Upsilon}$ expressing a mapping $\tilde{\Upsilon} : \acute{E} \to P(\mathcal{M})$ is then a neutrosophic soft set $(\tilde{\Upsilon}, \acute{E})$ over $\mathcal{M}$, where $\tilde{\Upsilon}$ is referred to as the approximate function of the NSS $(\tilde{\Upsilon}, \acute{E})$. Stated differently, the neutrosophic soft set can be expressed as a collection of ordered pairs:*

$$(\tilde{\Upsilon}, \acute{E}) = \left\{ \left( e, \langle s, T_{\tilde{\Upsilon}(e)}(s), I_{\tilde{\Upsilon}(e)}(s), F_{\tilde{\Upsilon}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*Here, $T_{\tilde{\Upsilon}(e)}(s)$, $I_{\tilde{\Upsilon}(e)}(s)$, and $F_{\tilde{\Upsilon}(e)}(s)$ are the truth-membership, indeterminacy-membership, and falsity-membership functions of $\tilde{\Upsilon}(e)$, respectively, and they all lie within the interval $[0, 1]$. The inequality*

$$0 \leq T_{\tilde{\Upsilon}(e)}(s) + I_{\tilde{\Upsilon}(e)}(s) + F_{\tilde{\Upsilon}(e)}(s) \leq 3$$

*is evident as the supremum of each $T$, $I$, and $F$ is 1 and the infimum is 0. This means that each value is a typical value between 0 and 1.*

**Definition 2.** *[9] Let $(\tilde{\Upsilon}, \acute{E})$ be a neutrosophic soft set. Then $(\tilde{\Upsilon}, \acute{E})^c$ is the complement of $(\tilde{\Upsilon}, \acute{E})$:*

$$(\tilde{\Upsilon}, \acute{E})^c = \left\{ \left( e, \langle s, F_{\tilde{\Upsilon}(e)}(s), 1 - I_{\tilde{\Upsilon}(e)}(s), T_{\tilde{\Upsilon}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*It is obvious that:*

$$\left( (\tilde{\Upsilon}, \acute{E})^c \right)^c = (\tilde{\Upsilon}, \acute{E}).$$

**Definition 3.** *[10] Let $(\tilde{\Upsilon}, \acute{E})$ and $(\tilde{\Lambda}, e)$ be two neutrosophic soft sets. Then $(\tilde{\Upsilon}, \acute{E})$ is said to be a neutrosophic soft subset of $(\tilde{\Lambda}, e)$ if:*

$$T_{\tilde{\Upsilon}(e)}(s) \leq T_{\tilde{\Lambda}(e)}(s), \quad I_{\tilde{\Upsilon}(e)}(s) \leq I_{\tilde{\Lambda}}(e)(s), \quad F_{\tilde{\Upsilon}(e)}(s) \geq F_{\tilde{\Lambda}(e)}(s), \quad \forall e \in \acute{E}, \forall s \in \mathcal{M}.$$

*It is denoted by:*

$$(\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\Lambda}, e).$$

**Definition 4.** *[10] Let $(\tilde{\Upsilon}_1, e)$ and $(\tilde{\Upsilon}_2, e)$ be two neutrosophic soft sets. Then their union is represented by $(\tilde{\Upsilon}_1, e) \Cup (\tilde{\Upsilon}_2, e) = (\tilde{\Upsilon}_3, e)$ as:*

$$(\tilde{\Upsilon}_3, e) = \left\{ \left( e, \langle s, T_{\tilde{\Upsilon}_{3(e)}}(s), I_{\tilde{\Upsilon}_{3(e)}}(s), F_{\tilde{\Upsilon}_{3(e)}}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*Where:*

$$T_{\tilde{\Upsilon}_{3(e)}}(s) = \max\{T_{\tilde{\Upsilon}_{1(e)}}(s), T_{\tilde{\Upsilon}_{2(e)}}(s)\},$$
$$I_{\tilde{\Upsilon}_{3(e)}}(s) = \max\{I_{\tilde{\Upsilon}_{1(e)}}(s), I_{\tilde{\Upsilon}_{2(e)}}(s)\},$$
$$F_{\tilde{\Upsilon}_{3(e)}}(s) = \min\{F_{\tilde{\Upsilon}_{1(e)}}(s), F_{\tilde{\Upsilon}_{2(e)}}(s)\}.$$

**Definition 5.** *[10] Let $(\tilde{\Upsilon}_1, e)$ and $(\tilde{\Upsilon}_2, e)$ be two neutrosophic soft sets. Then their intersection is symbolized by $(\tilde{\Upsilon}_1, e) \Cap (\tilde{\Upsilon}_2, e) = (\tilde{\Upsilon}_3, e)$ as:*

$$(\tilde{\Upsilon}_3, e) = \left\{ \left( e, \langle s, T_{\tilde{\Upsilon}_{3(e)}}(s), I_{\tilde{\Upsilon}_{3(e)}}(s), F_{\tilde{\Upsilon}_{3(e)}}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*Where:*

$$T_{\tilde{\Upsilon}_{3(e)}}(s) = \min\{T_{\tilde{\Upsilon}_{1(e)}}(s), T_{\tilde{\Upsilon}_{2(e)}}(s)\},$$
$$I_{\tilde{\Upsilon}_{3(e)}}(s) = \min\{I_{\tilde{\Upsilon}_{1(e)}}(s), I_{\tilde{\Upsilon}_{2(e)}}(s)\},$$
$$F_{\tilde{\Upsilon}_{3(e)}}(s) = \max\{F_{\tilde{\Upsilon}_{1(e)}}(s), F_{\tilde{\Upsilon}_{2(e)}}(s)\}.$$

**Definition 6.** *[10] Let $(\tilde{\Upsilon}_1, e)$ and $(\tilde{\Upsilon}_2, e)$ be two neutrosophic soft sets. Then the difference operation on them is denoted by $(\tilde{\Upsilon}_1, e) \setminus (\tilde{\Upsilon}_2, e) = (\tilde{\Upsilon}_3, e)$ and is defined by:*

$$(\tilde{\Upsilon}_3, e) = \left\{ \left( e, \langle s, T_{\tilde{\Upsilon}_{3(e)}}(s), I_{\tilde{\Upsilon}_{3(e)}}(s), F_{\tilde{\Upsilon}_{3(e)}}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*Where:*

$$T_{\tilde{\Upsilon}_{3(e)}}(s) = \min\{T_{\tilde{\Upsilon}_{1(e)}}(s), T_{\tilde{\Upsilon}_{2(e)}}(s)\},$$
$$I_{\tilde{\Upsilon}_{3(e)}}(s) = \min\{I_{\tilde{\Upsilon}_{1(e)}}(s), I_{\tilde{\Upsilon}_{2(e)}}(s)\},$$
$$F_{\tilde{\Upsilon}_{3(e)}}(s) = \max\{F_{\tilde{\Upsilon}_{1(e)}}(s), F_{\tilde{\Upsilon}_{2(e)}}(s)\}.$$

**Definition 7.** *[10] 1. A neutrosophic soft set $(\tilde{\Upsilon}, \acute{E})$ is said to be a null neutrosophic soft set if:*

$$T_{\tilde{\Upsilon}(e)}(s) = 0, \quad I_{\tilde{\Upsilon}(e)}(s) = 0, \quad F_{\tilde{\Upsilon}(e)}(s) = 1, \quad \forall e \in \acute{E}, \forall s \in \mathcal{M}.$$

*It is denoted by $0_{(\mathcal{M}, \acute{E})}$.*

*2. A neutrosophic soft set $(\tilde{\Upsilon}, \acute{E})$ is said to be an absolute neutrosophic soft set if:*

$$T_{\tilde{\Upsilon}(e)}(s) = 1, \quad I_{\tilde{\Upsilon}(e)}(s) = 1, \quad F_{\tilde{\Upsilon}(e)}(s) = 0, \quad \forall e \in \acute{E}, \forall s \in \mathcal{M}.$$

*It is symbolized as $1(\mathcal{M}, \acute{E})$.*

*It is obvious that:*

$$0_{(\mathcal{M}, \acute{E})} = 1_{(\mathcal{M}, \acute{E})}, \quad 1_{(\mathcal{M}, \acute{E})} = 0_{(\mathcal{M}, \acute{E})}.$$

## 3. A New Approach to Operations on Quadri-Partitioned Neutrosophic Soft Sets

Neutrosophic set theory (NST), a generality of vague set theory (VST), is regarded as the most appealing theory since it considers the three possible membership values: true, false, and indeterminacy. The principles are all quite obvious, but the third one is particularly fascinating since it addresses uncertainty, which arises in all aspects of daily life. One can make the situation more certain and free of error if the indeterminacy membership is refined. This can be accomplished by breaking down the indeterminacy into two distinct categories of possible values: 'relative true' and 'relative false,' each representing different aspects of the uncertainty. This section is devoted to the most basic operations of union, intersection, difference, and absolute null, absolute HPNNNs. Theorems and examples are given for better understanding the situation.

**Definition 8.** *Let e be the set of parameters and* $\mathcal{M}$ *be the key set. Let* $P(\mathcal{M})$ *represent the power set of* $\mathcal{M}$*. Then, a quadri-partitioned neutrosophic soft set* $(\tilde{F}, \acute{E})$ *over* $\mathcal{M}$ *is a mapping* $\tilde{F} : e \to P(\mathcal{M})$*, where* $\tilde{F}$ *is the function of the quadri-partitioned neutrosophic soft set* $(\tilde{F}, \acute{E})$*. Symbolically,*

$$(\tilde{F}, \acute{E}) = \left\{ \left( e, \langle s, AbT_{\tilde{F}(e)}(s), ReT_{\tilde{F}(e)}(s), ReF_{\tilde{F}(e)}(s), AbF_{\tilde{F}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*Here,* $AbT_{\tilde{F}(e)}(s)$*,* $ReT_{\tilde{F}(e)}(s)$*,* $ReF_{\tilde{F}(e)}(s)$*, and* $AbF_{\tilde{F}(e)}(s)$ *belong to the interval* $[0, 1]$*. Respectively, these functions are called the absolute true-membership, relative true-membership, relative false-membership, and absolute false-membership functions of* $\tilde{F}(e)$*. Since the supremum of each function is 1 and the infimum is 0, the following inequality holds:*

$$0 \leq AbT_{\tilde{F}(e)}(s) + ReT_{\tilde{F}(e)}(s) + ReF_{\tilde{F}(e)}(s) + AbF_{\tilde{F}(e)}(s) \leq 4.$$

**Definition 9.** *Let* $(\tilde{F}, \acute{E})$ *be a quadri-partitioned neutrosophic soft set over the key set* $\mathcal{M}$*. Then, the complement of* $(\tilde{F}, \acute{E})$ *is denoted by* $(\tilde{F}, \acute{E})^c$ *and is defined as follows:*

$$(\tilde{F}, \acute{E})^c = \left\{ \left( e, \langle s, AbF_{\tilde{F}(e)}(s), ReF_{\tilde{F}(e)}(s), ReT_{\tilde{F}(e)}(s), AbT_{\tilde{F}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*It follows that* $\left( (\tilde{F}, \acute{E})^c \right)^c = (\tilde{F}, \acute{E})$*.*

**Definition 10.** *Let* $(\tilde{F}, \acute{E})$ *and* $(\tilde{G}, \acute{E})$ *be two quadri-partitioned neutrosophic soft sets over the key set* $\mathcal{M}$*. Then,* $(\tilde{F}, \acute{E}) \subseteq (\tilde{G}, \acute{E})$ *if*

$$\begin{aligned} AbT_{\tilde{F}(e)}(s) &\preceq AbT_{\tilde{G}(e)}(s), \\ ReT_{\tilde{F}(e)}(s) &\preceq ReT_{\tilde{G}(e)}(s), \\ ReF_{\tilde{F}(e)}(s) &\succeq ReF_{\tilde{G}(e)}(s), \\ AbF_{\tilde{F}(e)}(s) &\succeq AbF_{\tilde{G}(e)}(s), \end{aligned}$$

*for all* $e \in \acute{E}$ *and* $s \in \mathcal{M}$*. If* $(\tilde{F}, \acute{E}) \subseteq (\tilde{G}, \acute{E})$ *and* $(\tilde{F}, \acute{E}) \supseteq (\tilde{G}, \acute{E})$*, then* $(\tilde{F}, \acute{E}) = (\tilde{G}, \acute{E})$*.*

**Definition 11.** *Let* $(\tilde{F}, \acute{E})$ *and* $(\tilde{G}, \acute{E})$ *be two quadri-partitioned neutrosophic soft sets over the key set* $\mathcal{M}$ *such that* $(\tilde{F}, \acute{E}) \neq (\tilde{G}, \acute{E})$*. Then their union is denoted by* $(\tilde{F}, \acute{E}) \Cup (\tilde{G}, \acute{E}) = (\tilde{H}, \acute{E})$ *and is defined as:*

$$(\tilde{H}, e) = \left\{ \left( e, \langle s, AbT_{\tilde{H}(e)}(s), ReT_{\tilde{H}(e)}(s), ReF_{\tilde{H}(e)}(s), AbF_{\tilde{H}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in E \right\}.$$

*where*

$$\begin{aligned} AbT_{\tilde{H}(e)}(s) &= \max \left\{ AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s) \right\}, \\ ReT_{\tilde{H}(e)}(s) &= \max \left\{ ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s) \right\}, \\ ReF_{\tilde{H}(e)}(s) &= \min \left\{ ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s) \right\}, \\ AbF_{\tilde{H}(e)}(s) &= \min \left\{ AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s) \right\}. \end{aligned}$$

**Definition 12.** *Let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be two quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$ such that $(\tilde{F}, \acute{E}) \neq (\tilde{G}, \acute{E})$. Then their intersection is denoted by $(\tilde{F}, \acute{E}) \Cap (\tilde{G}, \acute{E}) = (\tilde{H}, \acute{E})$ and is defined as:*

$$(\tilde{H}, \acute{E}) = \left\{ \left( e, \langle s, AbT_{\tilde{H}(e)}(s), ReT_{\tilde{H}(e)}(s), ReF_{\tilde{H}(e)}(s), AbF_{\tilde{H}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*where*

$$AbT_{\tilde{H}(e)}(s) = \min \left\{ AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s) \right\},$$
$$ReT_{\tilde{H}(e)}(s) = \min \left\{ ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s) \right\},$$
$$ReF_{\tilde{H}(e)}(s) = \max \left\{ ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s) \right\},$$
$$AbF_{\tilde{H}(e)}(s) = \max \left\{ AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s) \right\}.$$

**Definition 13.** *Let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be two quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$ such that $(\tilde{F}, \acute{E}) \neq (\tilde{G}, \acute{E})$. Then, their difference is denoted by $(\tilde{H}, \acute{E}) = (\tilde{F}, \acute{E}) \setminus (\tilde{G}, \acute{E})$ and is defined as:*

$$(\tilde{H}, \acute{E}) = (\tilde{F}, \acute{E}) \Cap (\tilde{G}, \acute{E})^c,$$

*such that*

$$(\tilde{H}, \acute{E}) = \left\{ \left( e, \langle s, AbT_{\tilde{H}(e)}(s), ReT_{\tilde{H}(e)}(s), ReF_{\tilde{H}(e)}(s), AbF_{\tilde{H}(e)}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

*where*

$$AbT_{\tilde{H}(e)}(s) = \min \left\{ AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s) \right\},$$
$$ReT_{\tilde{H}(e)}(s) = \min \left\{ ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s) \right\},$$
$$ReF_{\tilde{H}(e)}(s) = \max \left\{ ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s) \right\},$$
$$AbF_{\tilde{H}(e)}(s) = \max \left\{ AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s) \right\}.$$

**Definition 14.** *Let $\{(\tilde{F}_i, \acute{E}) : i \in I\}$ be a family of quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$. Then, the union and intersection of this family are defined as follows:*

$$\bigcup_{i \in I} (\tilde{F}_i, \acute{E}) = \left\{ \left( e, \langle s, \sup_{i \in I} AbT_{\tilde{F}_{i(e)}}(s), \sup_{i \in I} ReT_{\tilde{F}_{i(e)}}(s), \inf_{i \in I} ReF_{\tilde{F}_{i(e)}}(s), \inf_{i \in I} AbF_{\tilde{F}_{i(e)}}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

$$\bigcap_{i \in I} (\tilde{F}_i, \acute{E}) = \left\{ \left( e, \langle s, \inf_{i \in I} AbT_{\tilde{F}_{i(e)}}(s), \inf_{i \in I} ReT_{\tilde{F}_{i(e)}}(s), \sup_{i \in I} ReF_{\tilde{F}_{i(e)}}(s), \sup_{i \in I} AbF_{\tilde{F}_{i(e)}}(s) \rangle \right) : s \in \mathcal{M}, e \in \acute{E} \right\}.$$

**Definition 15.** *Let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be two quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$. Then, the "AND" operation on them is denoted by $(\tilde{F}, \acute{E}) \wedge (\tilde{G}, \acute{E}) = (\tilde{H}, \acute{E} \times e)$ and is defined as:*

$$(\tilde{H}, \acute{E} \times e) = \{ ((e_1, e_2), \langle s, AbT_{\tilde{H}}(e_1, e_2)(s), ReT_{\tilde{H}}(e_1, e_2)(s), ReF_{\tilde{H}}(e_1, e_2)(s), AbF_{\tilde{H}}(e_1, e_2)(s) \rangle) : (e_1, e_2) \in e \times e$$

*where*

$$AbT_{\tilde{H}(e)}(s) = \min \left\{ AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s) \right\},$$

$$ReT_{\tilde{H}(e)}(s) = \min\left\{ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s)\right\},$$
$$ReF_{\tilde{H}(e)}(s) = \max\left\{ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s)\right\},$$
$$AbF_{\tilde{H}(e)}(s) = \max\left\{AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s)\right\}.$$

**Definition 16.** *Let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be two quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$. Then, the "OR" operation on them is denoted by $(\tilde{F}, \acute{E}) \vee (\tilde{G}, \acute{E}) = (\tilde{H}, \acute{E} \times e)$ and is defined as:*

$$(\tilde{H}, \acute{E} \times e) = \Bigg\{\Big((e_1, e_2), \langle s, AbT_{\tilde{H}(e_1,e_2)}(s), ReT_{\tilde{H}(e_1,e_2)}(s),$$
$$ReF_{\tilde{H}(e_1,e_2)}(s), AbF_{\tilde{H}(e_1,e_2)}(s)\rangle\Big) : (e_1, e_2) \in e \times e, s \in \mathcal{M}\Bigg\}.$$

*where*

$$AbT_{\tilde{H}(e)}(s) = \max\left\{AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s)\right\},$$
$$ReT_{\tilde{H}(e)}(s) = \max\left\{ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s)\right\},$$
$$ReF_{\tilde{H}(e)}(s) = \min\left\{ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s)\right\},$$
$$AbF_{\tilde{H}(e)}(s) = \min\left\{AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s)\right\}.$$

**Definition 17.** *The quadri-partitioned neutrosophic soft set $(\tilde{F}, \acute{E})$ over the key set $\mathcal{M}$ is said to be a null quadri-partitioned neutrosophic soft set if*

$$AbT_{\tilde{F}(e)}(s) = 0, \quad ReT_{\tilde{F}(e)}(s) = 0, \quad \forall e \in \acute{E}, \forall s \in \mathcal{M},$$

$$ReF_{\tilde{F}(e)}(s) = 1, \quad AbF_{\tilde{F}(e)}(s) = 1, \quad \forall e \in \acute{E}, \forall s \in \mathcal{M}.$$

*It is denoted as $0_{(\mathcal{M},\acute{E})}$.*

**Definition 18.** *A quadri-partitioned neutrosophic soft set $(\tilde{F}, \acute{E})$ over the key set $\mathcal{M}$ is called an absolute quadri-partitioned neutrosophic soft set if*

$$AbT_{\tilde{F}(e)}(s) = 1, \quad ReT_{\tilde{F}(e)}(s) = 1, \quad \forall e \in \acute{E}, \forall s \in \mathcal{M},$$

$$ReF_{\tilde{F}(e)}(s) = 0, \quad AbF_{\tilde{F}(e)}(s) = 0, \quad \forall e \in \acute{E}, \forall s \in \mathcal{M}.$$

*Clearly,*

$$0^c_{(\mathcal{M},\acute{E})} = 1(\mathcal{M}, \acute{E}), \quad 1^c_{(\mathcal{M},\acute{E})} = 0_{(\mathcal{M},\acute{E})}.$$

**Definition 19.** *The family of all quadri-partitioned neutrosophic soft sets over $\mathcal{M}$ is designated as $\mathbb{P}NSS(\mathcal{M})$. A QPNS point for every point $s \in \mathcal{M}$ is defined as*

$$s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}$$

*for $0 \preceq p_1, p_2, p_3, p_4 \preceq 1, e \in \acute{E}$, and is given by:*

$$s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}(\mathcal{Y}) = \begin{cases} \langle p_1, p_2, p_3, p_4\rangle, & \text{if } e = e' \text{ and } \mathcal{Y} = s, \\ (0, 0, 0, 1), & \text{if } e' \neq e \text{ or } \mathcal{Y} \neq s. \end{cases}$$

**Definition 20.** *Let $(\tilde{F}, \acute{E})$ be a quadri-partitioned neutrosophic soft set over the key set $\mathcal{M}$. A point*

$$s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in QPNSS(\tilde{F}, \acute{E})$$

*if*

$$p_1 \preceq AbT_{\tilde{F}(e)}(s), \quad p_2 \preceq ReT_{\tilde{F}(e)}(s), \quad p_3 \succeq ReF_{\tilde{F}(e)}(s), \quad p_4 \succeq AbF_{\tilde{F}(e)}(s).$$

**Theorem 1.** *Let $(\tilde{F}, \acute{E})$, $(\tilde{G}, \acute{E})$, and $(\tilde{H}, \acute{E})$ be quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$. Then, the following properties hold:*

1.
$$(\tilde{F}, \acute{E}) \Cup \left[(\tilde{G}, \acute{E}) \Cup (\tilde{H}, \acute{E})\right] = \left[(\tilde{F}, \acute{E}) \Cup (\tilde{G}, \acute{E})\right] \Cup (\tilde{H}, \acute{E})$$

2.
$$(\tilde{F}, \acute{E}) \Cap \left[(\tilde{G}, \acute{E}) \Cap (\tilde{H}, \acute{E})\right] = \left[(\tilde{F}, \acute{E}) \Cap (\tilde{G}, \acute{E})\right] \Cap (\tilde{H}, \acute{E})$$

3.
$$(\tilde{F}, \acute{E}) \Cup \left[(\tilde{G}, \acute{E}) \Cap (\tilde{H}, \acute{E})\right] = \left[(\tilde{F}, \acute{E}) \Cup (\tilde{G}, \acute{E})\right] \Cap \left[(\tilde{F}, \acute{E}) \Cup (\tilde{H}, \acute{E})\right]$$

4.
$$(\tilde{F}, \acute{E}) \Cap \left[(\tilde{G}, \acute{E}) \Cup (\tilde{H}, \acute{E})\right] = \left[(\tilde{F}, \acute{E}) \Cap (\tilde{G}, \acute{E})\right] \Cup \left[(\tilde{F}, \acute{E}) \Cap (\tilde{H}, \acute{E})\right]$$

5.
$$(\tilde{F}, \acute{E}) \Cup 0_{(\mathcal{M}, \acute{E})} = (\tilde{F}, \acute{E})$$

6.
$$(\tilde{F}, \acute{E}) \Cap 0_{(\mathcal{M}, \acute{E})} = 0_{(\mathcal{M}, \acute{E})}$$

7.
$$(\tilde{F}, \acute{E}) \Cup 1_{(\mathcal{M}, \acute{E})} = 1_{(\mathcal{M}, \acute{E})}$$

8.
$$(\tilde{F}, \acute{E}) \Cap 1_{(\mathcal{M}, \acute{E})} = (\tilde{F}, \acute{E})$$

*Proof.*

Given $(\tilde{F}, \acute{E})$, $(\tilde{G}, \acute{E})$, and $(\tilde{H}, \acute{E})$ are quadri-partitioned neutrosophic soft sets, then,

$$(\tilde{F}, \acute{E})\tilde{\vee}[(\tilde{G}, \acute{E})\tilde{\vee}(\tilde{H}, \acute{E})] = [(\tilde{F}, \acute{E})\tilde{\vee}(\tilde{G}, \acute{E})]\tilde{\vee}(\tilde{H}, \acute{E})$$

$$\Rightarrow \quad s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{F}, \acute{E})\tilde{\vee}[(\tilde{G}, \acute{E})\tilde{\vee}(\tilde{H}, \acute{E})]$$

$$\Rightarrow \quad s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{F}, \acute{E}) \quad \text{or} \quad s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in [(\tilde{G}, \acute{E})\tilde{\vee}(\tilde{H}, \acute{E})]$$

$$\Rightarrow \quad s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{F}, \acute{E})$$

$$s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in \Big(e, s, \Big(\max\Big[AbT_{\tilde{G}(e)}(s), AbT_{\tilde{H}(e)}(s)\Big], \max\Big[ReT_{\tilde{G}(e)}(s), ReT_{\tilde{H}(e)}(s)\Big]\Big),$$

$$\Big(\min\Big[ReF_{\tilde{G}(e)}(s), ReF_{\tilde{H}(e)}(s)\Big], \min\Big[AbF_{\tilde{G}(e)}(s), AbF_{\tilde{H}(e)}(s)\Big]\Big)\Big), \quad \forall e \in \acute{E}, s \in \mathcal{M}$$

$$\Rightarrow \quad s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in \Big(e, s, \Big(\max\Big[AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s)\Big], \max\Big[ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s)\Big]\Big),$$

$$\left(\min\left[ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s)\right], \min\left[AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s)\right]\right)\Big), \quad \forall e \in \acute{E}, s \in \mathcal{M}$$

$$\text{i.e. } s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in [(\tilde{F},\acute{E})\tilde{\vee}(\tilde{G},\acute{E})]\tilde{\vee}(\tilde{H},\acute{E})$$

$$\Rightarrow (\tilde{F},\acute{E})\tilde{\vee}[(\tilde{G},\acute{E})\tilde{\vee}(\tilde{H},\acute{E})] \subseteq [(\tilde{F},\acute{E})\tilde{\vee}(\tilde{G},\acute{E})]\tilde{\vee}(\tilde{H},\acute{E})$$

Similarly,

$$[(\tilde{F},\acute{E})\tilde{\vee}(\tilde{G},\acute{E})]\tilde{\vee}(\tilde{H},\acute{E}) \subseteq (\tilde{F},\acute{E})\tilde{\vee}[(\tilde{G},\acute{E})\tilde{\vee}(\tilde{H},\acute{E})]$$

$$\Rightarrow (\tilde{F},\acute{E})\tilde{\vee}[(\tilde{G},\acute{E})\tilde{\vee}(\tilde{H},\acute{E})] = [(\tilde{F},\acute{E})\tilde{\vee}(\tilde{G},\acute{E})]\tilde{\vee}(\tilde{H},\acute{E})$$

Thus, in a similar fashion, we can prove the rest of the results.

**Theorem 2.** *Let $(\tilde{F},\acute{E})$ and $(\tilde{G},\acute{E})$ be quadri-partitioned neutrosophic soft sets over the key set $\mathcal{M}$, then*

$$[(\tilde{F},\acute{E}) \Cup (\tilde{G},\acute{E})]^c = (\tilde{F},\acute{E})^c \Cap (\tilde{G},\acute{E})^c;$$
$$[(\tilde{F},\acute{E}) \Cap (\tilde{G},\acute{E})]^c = (\tilde{F},\acute{E})^c \Cup (\tilde{G},\acute{E})^c.$$

*Proof.* Given $(\tilde{F},\acute{E})$ and $(\tilde{G},\acute{E})$ be quadri-partitioned neutrosophic soft sets, then

$$\forall e \in \acute{E}, \forall s \in \mathcal{M}, \quad (\tilde{F},\acute{E})\Cup(\tilde{G},\acute{E}) = \Big\{\Big(s, \max\left[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)\right], \max\left[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)\right],$$

$$\min\left[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)\right], \min\left[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)\right]\Big)\Big\}.$$

$$\left[(\tilde{F},\acute{E}) \Cup (\tilde{G},\acute{E})\right]^c = \Big\{\Big(s, \min\left[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)\right], \min\left[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)\right],$$

$$\max\left[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)\right], \max\left[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)\right]\Big)\Big\}.$$

Now,

$$(\tilde{F},\acute{E})^c = \Big\{\Big\langle s, \mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{F}(e)}(s)\Big\rangle\Big\},$$

$$(\tilde{G},\acute{E})^c = \Big\{\Big\langle s, \mathrm{AbF}_{\tilde{G}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)\Big\rangle\Big\}.$$

Thus,

$$(\tilde{F},\acute{E})^c \Cap (\tilde{G},\acute{E})^c = \Big\{\Big(s, \min\left[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)\right], \min\left[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)\right],$$

$$\max\left[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)\right], \max\left[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)\right]\Big)\Big\}.$$

Therefore,

$$\left[(\tilde{F},\acute{E}) \Cup (\tilde{G},\acute{E})\right]^c = (\tilde{F},\acute{E})^c \Cap (\tilde{G},\acute{E})^c.$$

Similarly,

$$\forall e \in \acute{E}, \forall s \in \mathcal{M}, \quad (\tilde{F},\acute{E})\Cap(\tilde{G},\acute{E}) = \Big\{\Big(s, \min\left[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)\right], \min\left[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)\right],$$

$$\max\left[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)\right], \max\left[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)\right]\Big)\Big\}.$$

$$\left[(\tilde{F}, \acute{E}) \Cap (\tilde{G}, \acute{E})\right]^c = (\tilde{F}, \acute{E})^c \Cup (\tilde{G}, \acute{E})^c.$$

**Theorem 3.** *Let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be quadri-partitioned neutrosophic soft sets over key set $\mathcal{M}$. Then:*
*1. $[(\tilde{F}, \acute{E})\tilde{\vee}(\tilde{G}, \acute{E})]^c = (\tilde{F}, \acute{E})^c\tilde{\wedge}(\tilde{G}, \acute{E})^c$. 2. $[(\tilde{F}, \acute{E})\tilde{\wedge}(\tilde{G}, \acute{E})]^c = (\tilde{F}, \acute{E})^c\tilde{\vee}(\tilde{G}, \acute{E})^c$.*

*Proof.* 1. For all $(e_1, e_2) \in e \times e$ and for all $s \in \mathcal{M}$:

$$(\tilde{F}, \acute{E})\tilde{\vee}(\tilde{G}, \acute{E}) = \Big(s, \max[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)], \max[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)],$$

$$\min[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)], \min[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)]\Big).$$

Taking the complement:

$$[(\tilde{F}, \acute{E})\tilde{\vee}(\tilde{G}, \acute{E})]^c = \Big(s, \min[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)], \min[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)],$$

$$\max[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)], \max[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)]\Big).$$

By considering the individual complements:

$$(\tilde{F}, \acute{E})^c = \Big(s, \mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{F}(e)}(s)\Big),$$

$$(\tilde{G}, \acute{E})^c = \Big(s, \mathrm{AbF}_{\tilde{G}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)\Big).$$

Therefore,

$$(\tilde{F}, \acute{E})^c\tilde{\wedge}(\tilde{G}, \acute{E})^c = \Big(s, \min[\mathrm{AbF}_{\tilde{F}(e)}(s), \mathrm{AbF}_{\tilde{G}(e)}(s)], \min[\mathrm{ReF}_{\tilde{F}(e)}(s), \mathrm{ReF}_{\tilde{G}(e)}(s)],$$

$$\max[\mathrm{ReT}_{\tilde{F}(e)}(s), \mathrm{ReT}_{\tilde{G}(e)}(s)], \max[\mathrm{AbT}_{\tilde{F}(e)}(s), \mathrm{AbT}_{\tilde{G}(e)}(s)]\Big).$$

Hence, $[(\tilde{F}, \acute{E})\tilde{\vee}(\tilde{G}, \acute{E})]^c = (\tilde{F}, \acute{E})^c\tilde{\wedge}(\tilde{G}, \acute{E})^c$.
2. A similar approach can be used to prove:

$$[(\tilde{F}, \acute{E})\tilde{\wedge}(\tilde{G}, \acute{E})]^c = (\tilde{F}, \acute{E})^c\tilde{\vee}(\tilde{G}, \acute{E})^c.$$

**Example 1.** *Let $\mathcal{M} = \{s_1, s_2, s_3\}$ be the key set and the set of parameters $e = \{e_1, e_2\}$. Let us develop the quadri-partitioned neutrosophic soft sets $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ over the key set $\mathcal{M}$ as follows:*

$$(\tilde{F}, \acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{2}{10}, \frac{3}{10}, \frac{7}{10}, \frac{8}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{4}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \\ e_2 = \langle s_1, \frac{3}{10}, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{1}{10}, \frac{5}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{5}{10}\rangle \end{bmatrix}$$

*Let $\mathcal{M} = \{s_1, s_2, s_3\}$ be the key set and the set of parameters $e = \{e_1, e_2\}$. The quadri-partitioned neutrosophic soft set $(\tilde{G}, \acute{E})$ over the key set $\mathcal{M}$ is defined as:*

$$(\tilde{G}, \acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \langle s_3, \frac{3}{10}, \frac{5}{10}, \frac{6}{10}, \frac{2}{10}\rangle, \\ e_2 = \langle s_1, \frac{3}{10}, \frac{4}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_2, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{6}{10}, \frac{6}{10}, \frac{3}{10}\rangle \end{bmatrix}$$

*Then, their union, intersection, AND, and OR operations are given as follows:*

$$(\tilde{F},\acute{E}) \Cup (\tilde{G},\acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \langle s_3, \frac{3}{10}, \frac{5}{10}, \frac{6}{10}, \frac{2}{10}\rangle, \\ e_2 = \langle s_1, \frac{3}{10}, \frac{4}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_2, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{6}{10}, \frac{6}{10}, \frac{3}{10}\rangle \end{bmatrix}$$

$$(\tilde{F},\acute{E}) \Cap (\tilde{G},\acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{2}{10}, \frac{3}{10}, \frac{7}{10}, \frac{8}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{4}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \\ e_2 = \langle s_1, \frac{3}{10}, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{1}{10}, \frac{5}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{5}{10}\rangle \end{bmatrix}$$

*The AND and OR operations of the quadri-partitioned neutrosophic soft sets are defined as follows:*

$$(\tilde{F},\acute{E}) \wedge (\tilde{G},\acute{E}) = \begin{bmatrix} (e_1,e_1) = \langle s_1, \frac{2}{10}, \frac{3}{10}, \frac{7}{10}, \frac{8}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{4}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \\ (e_1,e_2) = \langle s_1, \frac{2}{10}, \frac{3}{10}, \frac{7}{10}, \frac{8}{10}\rangle, \langle s_2, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \\ (e_2,e_1) = \langle s_1, \frac{3}{10}, \frac{2}{10}, \frac{7}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{1}{10}, \frac{5}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_3, \frac{3}{10}, \frac{3}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \\ (e_2,e_2) = \langle s_1, \frac{3}{10}, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{1}{10}, \frac{5}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{5}{10}\rangle \end{bmatrix}$$

$$(\tilde{F},\acute{E}) \vee (\tilde{G},\acute{E}) = \begin{bmatrix} (e_1,e_1) = \langle s_1, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \langle s_3, \frac{3}{10}, \frac{5}{10}, \frac{6}{10}, \frac{2}{10}\rangle, \\ (e_1,e_2) = \langle s_1, \frac{2}{10}, \frac{3}{10}, \frac{7}{10}, \frac{8}{10}\rangle, \langle s_2, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \\ (e_2,e_1) = \langle s_1, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{6}{10}\rangle, \langle s_2, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{3}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{2}{10}\rangle, \\ (e_2,e_2) = \langle s_1, \frac{3}{10}, \frac{4}{10}, \frac{6}{10}, \frac{5}{10}\rangle, \langle s_2, \frac{2}{10}, \frac{6}{10}, \frac{6}{10}, \frac{4}{10}\rangle, \langle s_3, \frac{4}{10}, \frac{6}{10}, \frac{6}{10}, \frac{3}{10}\rangle \end{bmatrix}$$

## 4. A New Approach to Operations on Quadri-partitioned Neutrosophic Soft Topological Space

The notion of quadri-partitioned neutrosophic soft topological space is presented in this section. The terms QPNS semi-open, QPNS pre-open and QPNS $*_b$open sets are defined. One of these intriguing QPNS generalized open sets, referred to as the QPNS pre-open set, is selected, and certain fundamentals are then produced based on this description. These consist of the QPNS closer, QPNS exterior, QPNS boundary, and QPNS interior.

**Definition 21.** *Let the quadri-partitioned neutrosophic soft set $(\tilde{\mathcal{M}},\acute{E})$ be the family of all quadri-partitioned neutrosophic soft sets, and let $\tau \subset QPNSS(\tilde{\mathcal{M}},\acute{E})$. Then, $\tau$ is a quadri-partitioned neutrosophic soft topology (QPNST) on $\tilde{\mathcal{M}}$ if:*

*(i)* $0_{(\langle\mathcal{M}\rangle,\acute{E})}, 1_{(\langle\mathcal{M}\rangle,\acute{E})} \in \tau$,

*(ii) The union of any number of quadri-partitioned neutrosophic soft sets in $\tau$ is in $\tau$,*

*(iii) The intersection of a finite number of quadri-partitioned neutrosophic soft sets in $\tau$ is in $\tau$.*

*Then, $(\tilde{\mathcal{M}},\tau,\acute{E})$ is said to be a quadri-partitioned neutrosophic soft topological space (QPNSTS) over $\tilde{\mathcal{M}}$.*

**Definition 22.** *A quadri-partitioned neutrosophic soft topological space $(\tilde{\mathcal{M}},\tau,\acute{E})$ over $\tilde{\mathcal{M}}$ is denoted as QPNSTS. A quadri-partitioned neutrosophic soft set $(\tilde{F},\acute{E})$ is a QPNS neighborhood of a QPNS point $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{F},\acute{E})$, if there exists a QPNS open set $(\tilde{G},\acute{E})$ such that $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{G},\acute{E})$.*

**Definition 23.** *Let $(\mathcal{M},\tau_1,\acute{E})$ and $(\mathcal{M},\tau_2,\acute{E})$ be two quadri-partitioned neutrosophic soft topological spaces. Then, $(\mathcal{M},\tau_1,\tau_2,\acute{E})$ is called a quadri-partitioned neutrosophic soft bitopological space (QPNSBTS). If $(\mathcal{M},\tau_1,\tau_2,\acute{E})$ is a quadri-partitioned neutrosophic soft topological space, a QPNSS subset $(\tilde{F},\acute{E})$ is open in $(\mathcal{M},\tau_1,\tau_2,\acute{E})$ if there exist a QPNSS open set $(\tilde{G},\acute{E}) \in \tau_1$ and a QPNSS open set $(\tilde{H},\acute{E}) \in \tau_2$ such that:*

$$(\tilde{F},\acute{E}) = (\tilde{G},\acute{E}) \Cup (\tilde{H},\acute{E}).$$

**Theorem 4.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft topological space. Then $\tau_1 \Cap \tau_2$ is a quadri-partitioned neutrosophic soft topological space on $\mathcal{M}$.*

*Proof.* The first and third requirements are clear, and we move forward as follows for the second condition. Let $\{(\tilde{\Upsilon}_i, \acute{E}); i \in I\} \in \tau_1 \Cap \tau_2$. Then $(\tilde{\Upsilon}_i, \acute{E}) \in \tau_1$ and $(\tilde{\Upsilon}_i, \acute{E}) \in \tau_2$. Since $\tau_1$ and $\tau_2$ are quadri-partitioned neutrosophic soft topological spaces on $\mathcal{M}$, it follows that:

$$\bigcup_{i \in I} (\tilde{\Upsilon}_i, \acute{E}) \in \tau_1, \quad \bigcup_{i \in I} (\tilde{\Upsilon}_i, \acute{E}) \in \tau_2.$$

Thus, we conclude that:

$$\bigcup_{i \in I} (\tilde{\Upsilon}_i, \acute{E}) \in \tau_1 \Cap \tau_2.$$

**Remark 1.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft topological space. Then $\tau_1 \Cup \tau_2$ need not be a quadri-partitioned neutrosophic soft topological space on $\mathcal{M}$. This is illustrated in the following Example 2.*

**Example 2.** *Let $\mathcal{M} = \{s_1, s_2, s_3\}$, $\acute{E} = \{e_1, e_2\}$, and define the topologies:*

$$\tau_1 = \{0(\mathcal{M}, \acute{E}), 1(\mathcal{M}, \acute{E}), (\tilde{\Upsilon}, \acute{E}), (\tilde{\Xi}, \acute{E}), (\tilde{\Lambda}, \acute{E})\},$$
$$\tau_2 = \{0(\mathcal{M}, \acute{E}), 1(\mathcal{M}, \acute{E}), (\tilde{I}, \acute{E}), (\tilde{J}, \acute{E})\}.$$

*where $(\tilde{\Upsilon}, \acute{E})$, $(\tilde{\Xi}, \acute{E})$, $(\tilde{\Lambda}, \acute{E})$, $(\tilde{I}, \acute{E})$, and $(\tilde{J}, \acute{E})$ are quadri-partitioned neutrosophic soft subsets defined as follows:*

$$(\tilde{\Upsilon}, \acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{2}{10}, \frac{3}{10}, \frac{8}{10}, \frac{8}{10} \rangle, \\ \langle s_2, \frac{4}{10}, \frac{4}{10}, \frac{4}{10}, \frac{4}{10} \rangle, \\ \langle s_3, \frac{2}{10}, \frac{4}{10}, \frac{3}{10}, \frac{3}{10} \rangle; \\ e_2 = \langle s_1, \frac{3}{10}, \frac{2}{10}, \frac{6}{10}, \frac{6}{10} \rangle, \\ \langle s_2, \frac{1}{10}, \frac{5}{10}, \frac{5}{10}, \frac{5}{10} \rangle, \\ \langle s_3, \frac{4}{10}, \frac{3}{10}, \frac{5}{10}, \frac{5}{10} \rangle. \end{bmatrix}$$

$$(\tilde{J}, \acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{4}{10}, \frac{3}{10}, \frac{6}{10}, \frac{6}{10} \rangle, \langle s_2, \frac{4}{10}, \frac{5}{10}, \frac{3}{10}, \frac{3}{10} \rangle, \langle s_3, \frac{3}{10}, \frac{5}{10}, \frac{2}{10}, \frac{2}{10} \rangle, \\ e_2 = \langle s_1, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{5}{10} \rangle, \langle s_2, \frac{2}{10}, \frac{6}{10}, \frac{4}{10}, \frac{4}{10} \rangle, \langle s_3, \frac{4}{10}, \frac{6}{10}, \frac{3}{10}, \frac{3}{10} \rangle \end{bmatrix}.$$

$$(\tilde{L}, \acute{E}) = \begin{bmatrix} e_1 = \langle s_1, \frac{5}{10}, \frac{4}{10}, \frac{4}{10}, \frac{4}{10} \rangle, \langle s_2, \frac{6}{10}, \frac{6}{10}, \frac{2}{10}, \frac{2}{10} \rangle, \langle s_3, \frac{4}{10}, \frac{6}{10}, \frac{1}{10}, \frac{1}{10} \rangle, \\ e_2 = \langle s_1, \frac{4}{10}, \frac{6}{10}, \frac{3}{10}, \frac{3}{10} \rangle, \langle s_2, \frac{3}{10}, \frac{7}{10}, \frac{3}{10}, \frac{3}{10} \rangle, \langle s_3, \frac{5}{10}, \frac{7}{10}, \frac{1}{10}, \frac{1}{10} \rangle \end{bmatrix}.$$

$$(\tilde{I}, \acute{E}) = \Bigg[ e_1 = \langle s_1, \frac{6}{10}, \frac{6}{10}, \frac{2}{10}, \frac{2}{10} \rangle, \langle s_2, \frac{6}{10}, \frac{6}{10}, \frac{2}{10}, \frac{2}{10} \rangle, \langle s_3, \frac{4}{10}, \frac{6}{10}, \frac{1}{10}, \frac{1}{10} \rangle, e_2 = \langle s_1, \frac{5}{10}, \frac{6}{10}, \frac{2}{10}, \frac{2}{10} \rangle, \langle s_2, \frac{6}{10}, \frac{7}{10}, \frac{2}{10}, \frac{2}{10} \rangle, \langle s_3, \frac{5}{10}, \frac{5}{10}, \frac{1}{10}, \frac{1}{10} \rangle \Bigg].$$

$$(\tilde{J}, \acute{E}) = \Bigg[ e_1 = \langle s_1, \frac{01}{10}, \frac{02}{10}, \frac{07}{10}, \frac{07}{10} \rangle, \langle s_2, \frac{04}{10}, \frac{04}{10}, \frac{03}{10}, \frac{03}{10} \rangle, \langle s_3, \frac{02}{10}, \frac{04}{10}, \frac{02}{10}, \frac{02}{10} \rangle, e_2 = \langle s_1, \frac{03}{10}, \frac{02}{10}, \frac{05}{10}, \frac{05}{10} \rangle, \langle s_2, \frac{01}{10}, \frac{05}{10}, \frac{05}{10}, \frac{05}{10} \rangle, \langle s_3, \frac{04}{10}, \frac{03}{10}, \frac{05}{10}, \frac{05}{10} \rangle \Bigg].$$

*Here, $\tau_1 \Cup \tau_2 = \{0_{(\tilde{\mathcal{M}}, \acute{E})}, 1_{(\tilde{\mathcal{M}}, \acute{E})}, (\tilde{\Upsilon}, \acute{E}), (\tilde{\Xi}, \acute{E}), (\tilde{\Lambda}, \acute{E}), (\tilde{I}, \acute{E}), (\tilde{J}, \acute{E})\}$ is not a quadri-partitioned neutrosophic soft topological space on $\tilde{\mathcal{M}}$ as $(\tilde{\Lambda}, \acute{E}) \Cup (\tilde{I}, \acute{E})$ does not belong to $\tau_1 \Cup \tau_2$. This justifies that the union of two quadri-partitioned neutrosophic soft topological spaces is not necessarily a quadri-partitioned neutrosophic soft topological space.*

**Definition 24.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space. Then a quadri-partitioned neutrosophic soft set,*

$$(\tilde{\Lambda}, \acute{E}) = \left[ \left( e, \left\langle s, AbT_{\tilde{\Lambda}(e)}(s), ReT_{\tilde{\Lambda}(e)}(s), ReF_{\tilde{\Lambda}(e)}(s), AbF_{\tilde{\Lambda}(e)}(s) \right\rangle : s \in \mathcal{M} \right) : e \in \acute{E} \right]$$

*is a pairwise QPNS open set if there exist a QPNS open set $(\tilde{\Upsilon}, \acute{E})$ in $\tau_1$ and a QPNS open set $(\tilde{\Xi}, \acute{E})$ in $\tau_2$ such that for all $s \in \mathcal{M}$,*

$$(\tilde{\Lambda}, \acute{E}) = (\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E}) = \left\{ \left( e, \left\{ s, \begin{array}{l} AbT_{\tilde{H}(e)}(s) = \max[AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s)], \\ ReT_{\tilde{H}(e)}(s) = \max[ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s)], \\ ReF_{\tilde{H}(e)}(s) = \min[ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s)], \\ AbF_{\tilde{H}(e)}(s) = \min[AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s)] \end{array} \right\} \right) : e \in \acute{E} \right\}.$$

*This is denoted by $QPNSO(\mathcal{M}, \acute{E})$.*

**Definition 25.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space. Then a quadri-partitioned neutrosophic soft set,*

$$(\tilde{\Lambda}, \acute{E}) = \left[ \left( e, \left\langle \left\langle s, AbT_{\tilde{H}(e)}(s), ReT_{\tilde{H}(e)}(s), ReF_{\tilde{H}(e)}(s), AbF_{\tilde{\Lambda}(e)}(s) \right\rangle \right\rangle : s \in \mathcal{M} \right) : e \in \acute{E} \right]$$

*is called a pairwise QPNSCS if $(\tilde{\Lambda}, \acute{E})^c$ is a pairwise QPNSO. $(\tilde{\Lambda}, \acute{E})$ is a QPNS closed set if there exists a QPNS closed set $(\tilde{\Upsilon}, \acute{E})$ in $\tau_1$ and a QPNS closed set $(\tilde{\Xi}, \acute{E})$ in $\tau_2$ such that for all $s \in \mathcal{M}$,*

$$(\tilde{\Lambda}, \acute{E}) = (\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{\Xi}, \acute{E}) = \left\{ \left( e, \left\{ s, \begin{array}{l} AbT_{\tilde{H}(e)}(s) = \min[AbT_{\tilde{F}(e)}(s), AbT_{\tilde{G}(e)}(s)], \\ ReT_{\tilde{H}(e)}(s) = \min[ReT_{\tilde{F}(e)}(s), ReT_{\tilde{G}(e)}(s)], \\ ReF_{\tilde{H}(e)}(s) = \max[ReF_{\tilde{F}(e)}(s), ReF_{\tilde{G}(e)}(s)], \\ AbF_{\tilde{H}(e)}(s) = \max[AbF_{\tilde{F}(e)}(s), AbF_{\tilde{G}(e)}(s)] \end{array} \right\} \right) : e \in \acute{E} \right\}.$$

*This is denoted by $QPNSC(\mathcal{M}, \acute{E})$.*

**Definition 26.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$, and $(\tilde{\ddot{\Upsilon}}, \acute{E})$ be a QPNSS. Then:*

*(i)* $(\tilde{\ddot{\Upsilon}}, \acute{E})$ *is QPNS semi-open if* $(\tilde{\ddot{\Upsilon}}, \acute{E}) \subseteq NScl(NSint(\tilde{\ddot{\Upsilon}}, \acute{E}))$.

*(ii)* $(\tilde{\ddot{\Upsilon}}, \acute{E})$ *is QPNS pre-open (p-open) if* $(\tilde{\ddot{\Upsilon}}, \acute{E}) \subseteq NSint(NScl(\tilde{\ddot{\Upsilon}}, \acute{E}))$.

*(iii)* $(\tilde{\ddot{\Upsilon}}, \acute{E})$ *is QPNS $*b$ open if*

$$(\tilde{\ddot{\Upsilon}}, \acute{E}) \subseteq NScl(NSint(\tilde{\ddot{\Upsilon}}, \acute{E})) \Cup NSint(NScl(\tilde{\ddot{\Upsilon}}, \acute{E})),$$

*and QPNS $*_b$ close if*

$$(\tilde{\ddot{\Upsilon}}, \acute{E}) \supseteq NScl(NSint(\tilde{\ddot{\Upsilon}}, \acute{E})) \Cap NSint(NScl(\tilde{\ddot{\Upsilon}}, \acute{E})).$$

**Definition 27.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$, and $(\tilde{\ddot{\Upsilon}}, \acute{E})$ be a QPNS. The interior of $(\tilde{\ddot{\Upsilon}}, \acute{E})$, denoted by $(\tilde{\ddot{\Upsilon}}, \acute{E})^\circ$, is the union of all QPNS p-open sets of $(\tilde{\ddot{\Upsilon}}, \acute{E})$. Clearly, $(\tilde{\ddot{\Upsilon}}, \acute{E})^\circ$ is the largest QPNS p-open set contained in $(\tilde{\ddot{\Upsilon}}, \acute{E})$.*

**Definition 28.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space, and $(\tilde{\ddot{\Upsilon}}, \acute{E})$ be a QPNS. The frontier of $(\tilde{\ddot{\Upsilon}}, \acute{E})$, denoted by $Fr((\tilde{\ddot{\Upsilon}}, \acute{E}))$, is a QPNS point $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}$ such that every QPNS p-open set containing $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}$ contains at least one point of $(\tilde{\ddot{\Upsilon}}, \acute{E})$ and at least one QPNS point of $(\tilde{\ddot{\Upsilon}}, \acute{E})^c$.*

**Definition 29.** *If $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ is a quadri-partitioned neutrosophic soft bi-topological space and $(\tilde{\ddot{\Upsilon}}, \acute{E})$ is a QPNS, then the exterior of $(\tilde{\ddot{\Upsilon}}, \acute{E})$, denoted by $Ext((\tilde{\ddot{\Upsilon}}, \acute{E}))$, is a QPNS point $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}$ such that $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}$ is in the interior of $(\tilde{\ddot{\Upsilon}}, \acute{E})^c$, i.e., there exists a QPNS p-open set $(\tilde{g}, \acute{E})$ such that*

$$s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{g}, \acute{E}) \subseteq (\tilde{\ddot{\Upsilon}}, \acute{E})^c.$$

**Definition 30.** *If $(\tilde{\mathcal{M}}, \tau_1, \tau_2, \acute{E})$ and $(\langle Y\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \acute{E})$ are quadri-partitioned neutrosophic soft bi-topological spaces, and $(f, \tilde{\ddot{\Upsilon}}) : (\tilde{\mathcal{M}}, \tau_1, \tau_2, \acute{E}) \to (\langle Y\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \acute{E})$ is a QPNS mapping, then $(f, \tilde{\ddot{\Upsilon}})$ is said to be a QPNS p-close mapping if the image $(f, \tilde{\ddot{\Upsilon}})(\tilde{\ddot{\Upsilon}}, \acute{E})$ of each QPNS p-closed set $(\tilde{\ddot{\Upsilon}}, \acute{E})$ over $\tilde{\mathcal{M}}$ is a QPNS p-closed set in $\langle Y\rangle$.*

**Theorem 5.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$ and $(\tilde{\ddot{\Upsilon}}, \acute{E})$ be a QPNS subset. Then, $(\tilde{\ddot{\Upsilon}}, \acute{E})$ is a QPNS p-open set if and only if $(\tilde{\ddot{\Upsilon}}, \acute{E}) = (\tilde{\ddot{\Upsilon}}, \acute{E})^\circ$.*

*Proof.* Let $(\tilde{\ddot{\Upsilon}}, \acute{E})$ be a QPNS $p$-open set. Then, the largest QPNS $p$-open set contained in $(\tilde{\ddot{\Upsilon}}, \acute{E})$ is equal to $(\tilde{\ddot{\Upsilon}}, \acute{E})$. Hence, $(\tilde{\ddot{\Upsilon}}, \acute{E}) = (\tilde{\ddot{\Upsilon}}, \acute{E})^\circ$.

Conversely, it is known that $(\tilde{\ddot{\Upsilon}}, \acute{E})^\circ$ is a QPNS $p$-open set, and if $(\tilde{\ddot{\Upsilon}}, \acute{E}) = (\tilde{\ddot{\Upsilon}}, \acute{E})^\circ$, then $(\tilde{\ddot{\Upsilon}}, \acute{E})$ is a QPNS $p$-open set.

**Theorem 6.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$, and let $(\tilde{\Upsilon}, \acute{E})$ and $(\tilde{\Xi}, \acute{E})$ be QPNS subsets. Then:*

*(i)* $[(\tilde{\Upsilon}, \acute{E})^\circ]^\circ = (\tilde{\Upsilon}, \acute{E})^\circ$,

*(ii)* $(0_{(\tilde{\mathcal{M}},\acute{E})})^\circ = 0_{(\tilde{\mathcal{M}},\acute{E})}$ *and* $(1_{(\tilde{\mathcal{M}},\acute{E})})^\circ = 1_{(\tilde{\mathcal{M}},\acute{E})}$,

*(iii)* *If* $(\tilde{\Upsilon},\acute{E}) \subseteq (\tilde{\Xi},\acute{E})$, *then* $(\tilde{\Upsilon},\acute{E})^\circ \subseteq (\tilde{\Xi},\acute{E})^\circ$,

*(iv)* $[(\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E})]^\circ = (\tilde{\Upsilon},\acute{E})^\circ \Cap (\tilde{\Xi},\acute{E})^\circ$,

*(v)* $(\tilde{\Upsilon},\acute{E})^\circ \Cup (\tilde{\Xi},\acute{E})^\circ \subseteq [(\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Xi},\acute{E})]^\circ$.

*Proof.*

(i) Since $(\tilde{\Upsilon},\acute{E})^\circ$ is a QPNS $p$-open set, it follows that $[(\tilde{\Upsilon},\acute{E})^\circ]^\circ = (\tilde{\Upsilon},\acute{E})^\circ$.

(ii) Since $0_{(\tilde{\mathcal{M}},\acute{E})}$ and $1_{(\tilde{\mathcal{M}},\acute{E})}$ are always QPNS $p$-open sets, we have:

$$(0_{(\tilde{\mathcal{M}},\acute{E})})^\circ = 0_{(\tilde{\mathcal{M}},\acute{E})}, \quad \text{and} \quad (1_{(\tilde{\mathcal{M}},\acute{E})})^\circ = 1_{(\tilde{\mathcal{M}},\acute{E})}.$$

(iii) Given that $(\tilde{\Upsilon},\acute{E})^\circ \subseteq (\tilde{\Upsilon},\acute{E}) \subseteq (\tilde{\Xi},\acute{E})$ and $(\tilde{\Xi},\acute{E})^\circ \subseteq (\tilde{\Xi},\acute{E})$, we conclude that $(\tilde{\Upsilon},\acute{E})^\circ \subseteq (\tilde{\Xi},\acute{E})^\circ$.

(iv) Since $(\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E}) \subseteq (\tilde{\Upsilon},\acute{E})$ and $(\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E}) \subseteq (\tilde{\Xi},\acute{E})$, we have:

$$[(\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E})]^\circ \subseteq (\tilde{\Upsilon},\acute{E})^\circ \Cap (\tilde{\Xi},\acute{E})^\circ.$$

Conversely, since $(\tilde{\Upsilon},\acute{E})^\circ \subseteq (\tilde{\Upsilon},\acute{E})$ and $(\tilde{\Xi},\acute{E})^\circ \subseteq (\tilde{\Xi},\acute{E})$, we also obtain:

$$(\tilde{\Upsilon},\acute{E})^\circ \Cap (\tilde{\Xi},\acute{E})^\circ \subseteq (\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E}).$$

Since $(\tilde{\Upsilon},\acute{E})^\circ \Cap (\tilde{\Xi},\acute{E})^\circ$ is the largest QPNS $p$-open set contained in $(\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E})$, we conclude:

$$[(\tilde{\Upsilon},\acute{E}) \Cap (\tilde{\Xi},\acute{E})]^\circ = (\tilde{\Upsilon},\acute{E})^\circ \Cap (\tilde{\Xi},\acute{E})^\circ.$$

(v) Since $(\tilde{\Upsilon},\acute{E}) \subseteq (\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Xi},\acute{E})$ and $(\tilde{\Xi},\acute{E}) \subseteq (\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Xi},\acute{E})$, it follows that:

$$(\tilde{\Upsilon},\acute{E})^\circ \subseteq [(\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Xi},\acute{E})]^\circ, \quad \text{and} \quad (\tilde{\Xi},\acute{E})^\circ \subseteq [(\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Xi},\acute{E})]^\circ.$$

Thus,

$$(\tilde{\Upsilon},\acute{E})^\circ \Cup (\tilde{\Xi},\acute{E})^\circ \subseteq [(\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Xi},\acute{E})]^\circ.$$

**Theorem 7.** *Let* $(\mathcal{M},\tau_1,\tau_2,\acute{E})$ *be a quadri-partitioned neutrosophic soft bi-topological space over* $\mathcal{M}$, *and if* $(\tilde{\Upsilon},\acute{E})$ *is a QPNS subset, then* $(\tilde{\Upsilon},\acute{E})$ *is a QPNS p-closed set if and only if* $(\tilde{\Upsilon},\acute{E}) = \overline{(\tilde{\Upsilon},\acute{E})}$.

*Proof.* Suppose that $(\tilde{\Upsilon},\acute{E})$ is a QPNS p-closed set, then we have:

$$(\tilde{\Upsilon},\acute{E})^d = (\tilde{\Upsilon},\acute{E})$$

which implies that

$$(\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Upsilon},\acute{E})^d = (\tilde{\Upsilon},\acute{E})$$

Thus,

$$\overline{(\tilde{\Upsilon},\acute{E})} = (\tilde{\Upsilon},\acute{E})$$

Conversely, if $\overline{(\tilde{\Upsilon},\acute{E})} = (\tilde{\Upsilon},\acute{E})$, then

$$(\tilde{\Upsilon},\acute{E}) \Cup (\tilde{\Upsilon},\acute{E})^d = (\tilde{\Upsilon},\acute{E})$$

which implies that

$$(\tilde{\Upsilon},\acute{E})^d = (\tilde{\Upsilon},\acute{E})$$

Thus, $(\tilde{\Upsilon},\acute{E})$ is a QPNS p-closed set.

**Theorem 8.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$, and let $(\tilde{\Upsilon}, \acute{E})$ and $(\tilde{\Xi}, \acute{E})$ be QPNS subsets. Then:*

*(i)* $\overline{[\overline{(\tilde{\Upsilon}, \acute{E})}]} = \overline{(\tilde{\Upsilon}, \acute{E})}$,

*(ii)* $\overline{0_{(\langle\mathcal{M}\rangle, \acute{E})}} = 0_{(\langle\mathcal{M}\rangle, \acute{E})}$ *and* $\overline{1_{(\langle\mathcal{M}\rangle, \acute{E})}} = 1_{(\langle\mathcal{M}\rangle, \acute{E})}$,

*(iii)* *If* $(\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\Xi}, \acute{E})$, *then* $\overline{(\tilde{\Upsilon}, \acute{E})} \subseteq \overline{(\tilde{\Xi}, \acute{E})}$,

*(iv)* $\overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})} = \overline{(\tilde{\Upsilon}, \acute{E})} \Cup \overline{(\tilde{\Xi}, \acute{E})}$,

*(v)* $\overline{(\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{\Xi}, \acute{E})} \subseteq \overline{(\tilde{\Upsilon}, \acute{E})} \Cap \overline{(\tilde{\Xi}, \acute{E})}$.

*Proof.*

(i) If $\overline{(\tilde{\Upsilon}, \acute{E})} = \overline{(\tilde{\Xi}, \acute{E})}$, then $(\tilde{\Xi}, \acute{E})$ is a QPNS $p$-closed set. Hence, if $(\tilde{\Xi}, \acute{E})$ and its closure are equal, then $\overline{\overline{(\tilde{\Upsilon}, \acute{E})}} = \overline{(\tilde{\Upsilon}, \acute{E})}$.

(ii) Since $0_{(\langle\mathcal{M}\rangle, \acute{E})}$ and $1_{(\langle\mathcal{M}\rangle, \acute{E})}$ are always QPNS $p$-closed sets, it follows that $\overline{0_{(\langle\mathcal{M}\rangle, \acute{E})}} = 0_{(\langle\mathcal{M}\rangle, \acute{E})}$ and $\overline{1_{(\langle\mathcal{M}\rangle, \acute{E})}} = 1_{(\langle\mathcal{M}\rangle, \acute{E})}$.

(iii) Given $(\tilde{\Upsilon}, \acute{E}) \subseteq \overline{(\tilde{\Upsilon}, \acute{E})}$ and $(\tilde{\Xi}, \acute{E}) \subseteq \overline{(\tilde{\Xi}, \acute{E})}$, we obtain $(\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\Xi}, \acute{E}) \subseteq \overline{(\tilde{\Xi}, \acute{E})}$. Since $\overline{(\tilde{\Upsilon}, \acute{E})}$ is the smallest QPNS $p$-closed set covering $(\tilde{\Upsilon}, \acute{E})$, it follows that $\overline{(\tilde{\Upsilon}, \acute{E})} \subseteq \overline{(\tilde{\Xi}, \acute{E})}$.

(iv) Since $(\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})$ and $(\tilde{\Xi}, \acute{E}) \subseteq 0_{(\langle\mathcal{M}\rangle, \acute{E})} \Cup (\tilde{\Xi}, \acute{E})$, we obtain $\overline{(\tilde{\Upsilon}, \acute{E})} \subseteq \overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})}$ and $\overline{(\tilde{\Xi}, \acute{E})} \subseteq \overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})}$.

Thus, $\overline{(\tilde{\Upsilon}, \acute{E})} \Cup \overline{(\tilde{\Xi}, \acute{E})} \subseteq \overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})}$. Conversely, since $(\tilde{\Upsilon}, \acute{E}) \subseteq \overline{(\tilde{\Upsilon}, \acute{E})}$ and $(\tilde{\Xi}, \acute{E}) \subseteq \overline{(\tilde{\Xi}, \acute{E})}$, we conclude that $(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E}) \subseteq \overline{(\tilde{\Upsilon}, \acute{E})} \Cup \overline{(\tilde{\Xi}, \acute{E})}$. Since $\overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})}$ is the smallest QPNS $p$-closed set enclosing $(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})$, it follows that $\overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Xi}, \acute{E})} \subseteq \overline{(\tilde{\Upsilon}, \acute{E})} \Cup \overline{(\tilde{\Xi}, \acute{E})}$.

(v) Using a similar argument, we obtain $\overline{(\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{\Xi}, \acute{E})} \subseteq \overline{(\tilde{\Upsilon}, \acute{E})} \Cap \overline{(\tilde{\Xi}, \acute{E})}$.

**Theorem 9.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$, and let $(\tilde{\Upsilon}, \acute{E})$ be a QPNSS. Then, the following hold:*

*(i)* $[\overline{(\tilde{\Upsilon}, \acute{E})}]^c = [(\tilde{\Upsilon}, \acute{E})^c]^\circ$,

*(ii)* $[(\tilde{\Upsilon}, \acute{E})^\circ]^c = \overline{[(\tilde{\Upsilon}, \acute{E})^c]}$.

*Proof.*

(i)

$$\begin{aligned}
\overline{(\tilde{\Upsilon}, \acute{E})} &= \Cap\{(\tilde{H}, \acute{E}) \in (\mathcal{M}, \tau_1, \tau_2, \acute{E})^c : (\tilde{H}, \acute{E}) \supseteq (\tilde{\Upsilon}, \acute{E})\} \\
&\Rightarrow [\overline{(\tilde{\Upsilon}, \acute{E})}]^c = \left[\Cap\{(\tilde{H}, \acute{E}) \in (\mathcal{M}, \tau_1, \tau_2, \acute{E})^c : (\tilde{H}, \acute{E}) \supseteq (\tilde{\Upsilon}, \acute{E})\}\right]^c \\
&= \Cup\{(\tilde{H}, \acute{E})^c \in (\mathcal{M}, \tau_1, \tau_2, \acute{E}) : (\tilde{H}, \acute{E})^c \subseteq (\tilde{\Upsilon}, \acute{E})^c\} \\
&= [(\tilde{\Upsilon}, \acute{E})^c]^\circ.
\end{aligned}$$

(ii)

$$
\begin{aligned}
(\tilde{\Upsilon}, \acute{E})^{\circ} &= \Cup\{(\tilde{H}, \acute{E}) \in (\mathcal{M}, \tau_1, \tau_2, \acute{E}) : (\tilde{H}, \acute{E}) \subseteq (\tilde{\Upsilon}, \acute{E})\} \\
&\Rightarrow [(\tilde{\Upsilon}, \acute{E})^{\circ}]^c = \left[\Cap\{(\tilde{H}, \acute{E}) \in (\mathcal{M}, \tau_1, \tau_2, \acute{E}) : (\tilde{H}, \acute{E}) \subseteq (\tilde{\Upsilon}, \acute{E})\}\right]^c \\
&= \Cap\{(\tilde{H}, \acute{E})^c \in (\mathcal{M}, \tau_1, \tau_2, \acute{E})^c : (\tilde{H}, \acute{E})^c \supseteq (\tilde{\Upsilon}, \acute{E})^c\} \\
&= \overline{[(\tilde{\Upsilon}, \acute{E})^c]}.
\end{aligned}
$$

**Theorem 10.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a quadri-partitioned neutrosophic soft bi-topological space over $\mathcal{M}$. If $(\tilde{\Upsilon}, \acute{E})$ and $(\tilde{H}, \acute{E})$ are QPNS subsets, then:*

*(i)* $\mathrm{Ext}((\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E})) = \mathrm{Ext}((\tilde{\Upsilon}, \acute{E})) \Cup \mathrm{Ext}((\tilde{H}, \acute{E}))$.

*(ii)* $\mathrm{Ext}((\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E})) \supseteq \mathrm{Ext}((\tilde{\Upsilon}, \acute{E})) \Cup \mathrm{Ext}((\tilde{H}, \acute{E}))$.

*(iii)* $\mathrm{Fr}((\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E})) \subseteq \mathrm{Fr}(\tilde{\Upsilon}, \acute{E}) \Cup \mathrm{Fr}(\tilde{H}, \acute{E})$.

*(iv)* $\mathrm{Fr}((\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E})) \subseteq \mathrm{Fr}(\tilde{\Upsilon}, \acute{E}) \Cup \mathrm{Fr}(\tilde{H}, \acute{E})$.

*Proof.*

(i) Since

$$
\begin{aligned}
\mathrm{Ext}((\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E})) &= (((\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E}))^{\mathcal{C}})^{\circ} \\
&= ((\tilde{\Upsilon}, \acute{E})^{\mathcal{C}} \Cap (\tilde{H}, \acute{E})^{\mathcal{C}})^{\circ} \\
&= ((\tilde{\Upsilon}, \acute{E})^{\mathcal{C}})^{\circ} \Cap ((\tilde{H}, \acute{E})^{\mathcal{C}})^{\circ} \\
&= \mathrm{Ext}((\tilde{\Upsilon}, \acute{E})) \Cap \mathrm{Ext}((\tilde{H}, \acute{E})).
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
\mathrm{Ext}((\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E})) &= (((\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E}))^{\mathcal{C}})^{\circ} \\
&= (((\tilde{\Upsilon}, \acute{E})^{\mathcal{C}} \Cup (\tilde{H}, \acute{E})^{\mathcal{C}}))^{\circ} \\
&\supseteq ((\tilde{\Upsilon}, \acute{E})^{\mathcal{C}})^{\circ} \Cup ((\tilde{H}, \acute{E})^{\mathcal{C}})^{\circ} \\
&= \mathrm{Ext}((\tilde{\Upsilon}, \acute{E})) \Cup \mathrm{Ext}((\tilde{H}, \acute{E})).
\end{aligned}
$$

(iii) For the frontier:

$$
\begin{aligned}
\mathrm{Fr}((\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E})) &= \overline{(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E})} \Cap \overline{((\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{H}, \acute{E}))^{\mathcal{C}}} \\
&= (\overline{(\tilde{\Upsilon}, \acute{E})} \Cup \overline{(\tilde{H}, \acute{E})}) \Cap (\overline{(\tilde{\Upsilon}, \acute{E})^{\mathcal{C}}} \Cap \overline{(\tilde{H}, \acute{E})^{\mathcal{C}}}) \\
&\subseteq \mathrm{Fr}((\tilde{\Upsilon}, \acute{E})) \Cup \mathrm{Fr}((\tilde{H}, \acute{E})).
\end{aligned}
$$

(iv) Similarly, for the intersection:

$$
\begin{aligned}
\mathrm{Fr}((\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E})) &= \overline{(\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E})} \Cap \overline{((\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{H}, \acute{E}))^{\mathcal{C}}} \\
&\subseteq (\overline{(\tilde{\Upsilon}, \acute{E})} \Cap \overline{(\tilde{H}, \acute{E})}) \Cap (\overline{(\tilde{\Upsilon}, \acute{E})^{\mathcal{C}}} \Cup \overline{(\tilde{H}, \acute{E})^{\mathcal{C}}}) \\
&= \mathrm{Fr}((\tilde{\Upsilon}, \acute{E})) \Cup \mathrm{Fr}((\tilde{H}, \acute{E})).
\end{aligned}
$$

## 5. Few results on QPNS P-Compactness

In the context of QPNSBTS and QPNS p-compact spaces, the idea of QPNS compactness is examined in this section. The concept of reducibility to finite sub-covers is one of the features of a QPNS cover that we define and study. Important theorems prove that the intersection of QPNS p-closed sets is non-empty and that a QPNS p-compact space preserves the finite intersection property among its QPNS p-closed sets. We also prove the existence of disjoint QPNS p-open sets separating disjoint QPNS p-compact subsets in a QPNS p-Hausdorff space and show that a QPNS p-closed subset of a QPNS p-compact space stays QPNS p-compact. The theoretical underpinning of QPNS spaces is improved by this work, which advances our knowledge of compactness in soft topological spaces.

**Definition 31.** *Let $(\tilde{\Upsilon}, \acute{E}) = \{(\tilde{\Upsilon}, \acute{E})_i\}$ be the class of subsets of $\mathcal{M}$ and $(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\mathcal{M}}$. If $(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}(\tilde{\Upsilon}, \acute{E})_i$, then the class $\{(\tilde{\Upsilon}, \acute{E})_i\}$ is called a QPNS cover of $(\tilde{\Theta}, \acute{E})$.*

*This cover is known as finite, countable, or QPNS p-open according to whether the members of the above class are finite, countable, or QPNS p-open, respectively.*

*Additionally, let:*

$$(\tilde{\Theta}, \acute{E})_1 = (\tilde{\Upsilon}, \acute{E})_1 \Cup (\tilde{\Upsilon}, \acute{E})_2 \Cup (\tilde{\Upsilon}, \acute{E})_3.$$

*If*

$$(\tilde{\Theta}, \acute{E})_1 = \{(\tilde{\Upsilon}, \acute{E})_i : i \in \tilde{I}\}$$

*and*

$$(\tilde{\Theta}, \acute{E})_2 = \{(\tilde{\Lambda}, \acute{E})_i : i \in \tilde{I}\}$$

*are two QPNS covers of a set $(\tilde{\Theta}, \acute{E})$, i.e.,*

$$(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}(\tilde{\Upsilon}, \acute{E})_i \quad \text{and} \quad (\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}(\tilde{\Lambda}, \acute{E})_i,$$

*such that every member of $(\tilde{\Theta}, \acute{E})_2$ is also a member of $(\tilde{\Theta}, \acute{E})_1$, then $(\tilde{\Theta}, \acute{E})_2$ is called the QPNS sub-cover of $(\tilde{\Theta}, \acute{E})_1$.*

**Definition 32.** *Let $(\tilde{\Theta}, \acute{E}) = \{(\tilde{\Upsilon}, \acute{E})_i : i \in \tilde{I}\}$ be a class of QPNS subsets of $\mathcal{M}$, and suppose $(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\mathcal{M}}$ such that*

$$(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}(\tilde{\Upsilon}, \acute{E})_i.$$

*Then, as mentioned above, $(\tilde{\Theta}, \acute{E}) = \{(\tilde{\Upsilon}, \acute{E})_i : i \in \tilde{I}\}$ is a QPNS p-cover.*

*If we can select a finite number of QPNS sets from the above class $(\tilde{\Theta}, \acute{E})$, that is, if we can have*

$$(\tilde{\Upsilon}, \acute{E})_{i_1}, (\tilde{\Upsilon}, \acute{E})_{i_2}, (\tilde{\Upsilon}, \acute{E})_{i_3}, \ldots, (\tilde{\Upsilon}, \acute{E})_{i_k} \quad \text{from} \quad (\tilde{\Theta}, \acute{E})$$

*such that*

$$(\tilde{\Theta}, \acute{E}) \subseteq (\tilde{\Upsilon}, \acute{E})_{i_1} \Cup (\tilde{\Upsilon}, \acute{E})_{i_2} \Cup (\tilde{\Upsilon}, \acute{E})_{i_3} \Cup \cdots \Cup (\tilde{\Upsilon}, \acute{E})_{i_k},$$

*or equivalently,*

$$(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}_{n=1}^{k} (\tilde{\Upsilon}, \acute{E})_{i_n},$$

*then we say that the cover $(\tilde{\Theta}, \acute{E}) = \{(\tilde{\Upsilon}, \acute{E})_i : i \in \tilde{I}\}$ is reducible to a finite QPNS sub-cover.*

**Definition 33.** *Let $(\mathcal{M}, \tau_1, \tau_2, \acute{E})$ be a QPNSBTS over $\mathcal{M}$, and let $(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\mathcal{M}}$. The set $(\tilde{\Theta}, \acute{E})$ is called QPNS compact if every QPNS p-open cover of $(\tilde{\Theta}, \acute{E})$ is reducible to a finite sub-cover.*

*If we can find a cover of $(\tilde{\Theta}, \acute{E})$ that cannot be reduced to a finite sub-cover, then we say that $(\tilde{\Theta}, \acute{E})$ is not QPNS p-compact.*

*Similarly, if every QPNS p-open cover of $\mathcal{M}$ is reducible to a finite sub-cover, then we say that $\mathcal{M}$ is QPNS compact. That is, if*

$$(\tilde{\Theta}, \acute{E}) = \{(\tilde{\Upsilon}, \acute{E})_i : i \in \tilde{I}\}$$

*and*

$$(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}_{i\in\tilde{I}}(\tilde{\Upsilon}, \acute{E})_i,$$

*then if we can write*

$$(\tilde{\Theta}, \acute{E}) \subseteq \tilde{\Cup}_{n=1}^{k}(\tilde{\Upsilon}, \acute{E})_{i_n},$$

*or equivalently,*

$$(\tilde{\Theta}, \acute{E}) \subseteq (\tilde{\Upsilon}, \acute{E})_{i_1} \Cup (\tilde{\Upsilon}, \acute{E})_{i_2} \Cup (\tilde{\Upsilon}, \acute{E})_{i_3} \Cup \cdots \Cup (\tilde{\Upsilon}, \acute{E})_{i_k},$$

*then $(\tilde{\Theta}, \acute{E})$ is QPNS compact.*

**Definition 34.** *If $\tilde{\mathcal{M}}$ is a QPNS, the collection of QPNS subsets of $\tilde{\mathcal{M}}$, say $\{(\tilde{\Theta}, \acute{E})_\alpha : \alpha \in I\}$, is said to have the finite intersection property if we can find a finite soft QPNS sub-collection*

$$\{(\tilde{\Theta}, \acute{E})_{\alpha_1}, (\tilde{\Theta}, \acute{E})_{\alpha_2}, (\tilde{\Theta}, \acute{E})_{\alpha_3}, \ldots, (\tilde{\Theta}, \acute{E})_{\alpha_n}\}$$

*such that*

$$(\tilde{\Theta}, \acute{E})_{\alpha_1} \tilde{\Cap} (\tilde{\Theta}, \acute{E})_{\alpha_2} \tilde{\Cap} (\tilde{\Theta}, \acute{E})_{\alpha_3} \tilde{\Cap} \ldots \tilde{\Cap} (\tilde{\Theta}, \acute{E})_{\alpha_n} \neq \tilde{\Phi}$$

*or equivalently,*

$$\bigcap_{i=1}^{n}(\tilde{\Theta}, \acute{E})_{\alpha_i} \neq \tilde{\Phi}.$$

**Theorem 11.** *Suppose $(\tilde{\mathcal{M}}, \tau_1, \tau_2, \acute{E})$ is QPNS p-compact if and only if each class of QPNS p-closed sets with the finite intersection property has a non-empty intersection.*

*Proof.* If $(\tilde{\mathcal{M}}, \tau_1, \tau_2, \acute{E})$ is QPNS p-compact, we need to show that each class of QPNS p-closed sets with the finite intersection property has a non-empty intersection.

Let $\{(\tilde{\Upsilon}, \acute{E})_\alpha : \alpha \in I\}$ be a class of QPNS p-closed sets in $\tilde{\mathcal{M}}$ satisfying the finite intersection property, i.e.,

$$\bigcap_{i=1}^{n}(\tilde{\Upsilon}, \acute{E})_{\alpha_i} \neq \tilde{\Phi}.$$

We prove this result by contradiction. Suppose

$$\bigcap_{i=1}^{n}(\tilde{\Upsilon}, \acute{E})_{\alpha_i} = \tilde{\Phi}.$$

This implies that

$$\left(\bigcap_{i=1}^{n}(\tilde{\Upsilon}, \acute{E})_{\alpha_i}\right)^c = \tilde{\Phi}^c.$$

which further leads to

$$\left(\bigcup_{i=1}^{n}(\tilde{\Upsilon}, \acute{E})_{\alpha_i}\right)^c = \tilde{\Phi}^c.$$

Thus, we get

$$\tilde{\mathcal{M}} = \left(\bigcup_{i=1}^{n}(\tilde{\Upsilon}, \acute{E})_{\alpha_i}\right)^c.$$

Since $\{(\tilde{\Upsilon}, \acute{E})_\alpha : \alpha \in I\}$ is a collection of QPNS p-closed sets, their complements $\{(\tilde{\Upsilon}, \acute{E})_\alpha^c\}$ form a QPNS p-open cover of $\tilde{\mathcal{M}}$.

But $\tilde{\mathcal{M}}$ is given to be QPNS p-compact, so the above QPNS p-open cover must be reducible to a finite sub-cover, meaning

$$\tilde{\mathcal{M}} = \bigcup_{k=1}^{n}(\tilde{\Upsilon}, \acute{E})_{i_k}^c.$$

This implies

$$\left(\bigcap_{i=1}^{n}(\tilde{\Upsilon},\acute{E})_{i_k}\right)^c = \tilde{\mathcal{M}}^c.$$

which further implies

$$\left[\left(\bigcap_{k=1}^{n}(\tilde{\Upsilon},\acute{E})_{i_k}\right)^c\right]^c = \tilde{\Phi}.$$

Thus,

$$\bigcap_{k=1}^{n}(\tilde{\Upsilon},\acute{E})_{i_k} = \tilde{\Phi},$$

which contradicts the finite intersection property. Hence,

$$\bigcap_{i=1}^{n}(\tilde{\Upsilon},\acute{E})_{\alpha_i} \neq \tilde{\Phi}.$$

Conversely, suppose each class of QPNS p-closed sets with the finite intersection property has a non-empty intersection. We now prove that $\tilde{\mathcal{M}}$ is QPNS p-compact using a contradiction.

Assume that $\tilde{\mathcal{M}}$ is not QPNS p-compact. Then, there exists at least one QPNS p-open cover of $\tilde{\mathcal{M}}$ that is not reducible to a finite sub-cover. Let this QPNS p-open cover be

$$\{(\tilde{\mathcal{J}},\acute{E})_i : i \in I\}.$$

Since this is a QPNS p-open cover of $\tilde{\mathcal{M}}$, we have

$$\tilde{\mathcal{M}} = \bigcup_{i\in I}(\tilde{\mathcal{J}},\acute{E})_i.$$

However, by assumption,

$$\tilde{\mathcal{M}} \neq \bigcup_{k=1}^{m}(\tilde{\mathcal{J}},\acute{E})_{i_k}.$$

This implies

$$\bigcap_{k=1}^{m}(\tilde{\mathcal{J}},\acute{E})^c_{i_k} \neq \tilde{\Phi}.$$

where $(\tilde{\mathcal{J}},\acute{E})^c_{i_k}$ are QPNS p-closed. But by hypothesis,

$$\bigcap_{i\in I}(\tilde{\mathcal{J}},\acute{E})^c_i \neq \tilde{\Phi}.$$

which contradicts our assumption. Therefore, $\tilde{\mathcal{M}}$ must be QPNS p-compact.

**Theorem 12.** *If $(\tilde{\mathcal{M}},\tau_1,\tau_2,\acute{E})$ is a QPNSTS and a QPNS p-closed subset of a QPNS p-compact space, then it is also QPNS p-compact.*

*Proof.* Suppose that $(\tilde{\mathcal{M}},\tau_1,\tau_2,\acute{E})$ is a QPNSTS such that it is QPNS p-compact and $(\tilde{\Upsilon},\acute{E})$ is a QPNS p-closed subset of $\tilde{\mathcal{M}}$. We aim to show that $(\tilde{\Upsilon},\acute{E})$ is also QPNS p-compact.

Let $\{(\tilde{\mathcal{J}},\acute{E})_i : i \in I\}$ be a QPNS p-open cover of $(\tilde{\Upsilon},\acute{E})$. That is,

$$(\tilde{\Upsilon},\acute{E}) \subseteq \bigcup_{i\in I}(\tilde{\mathcal{J}},\acute{E})_i.$$

Then, $(\tilde{\Upsilon},\acute{E})$ will be QPNS p-compact if this QPNS p-open cover is reducible to a finite QPNS sub-cover.

Since $\{(\tilde{\mathcal{J}}, \acute{E})_i : i \in I\}$ is a QPNS p-open cover of $(\tilde{\Upsilon}, \acute{E})$, we have:

$$(\tilde{\Upsilon}, \acute{E}) \subseteq \bigcup_{i \in I} (\tilde{\mathcal{J}}, \acute{E})_i.$$

Also, since $\tilde{\mathcal{M}}$ can be expressed as the union of $(\tilde{\Upsilon}, \acute{E})$ and its QPNS p-complement $(\tilde{\Upsilon}, \acute{E})^c$, we obtain:

$$\tilde{\mathcal{M}} = (\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Upsilon}, \acute{E})^c.$$

Thus, taking the union with $(\tilde{\Upsilon}, \acute{E})^c$ on both sides, we get:

$$(\tilde{\Upsilon}, \acute{E}) \Cup (\tilde{\Upsilon}, \acute{E})^c \subseteq \left( \bigcup_{i \in I} (\tilde{\mathcal{J}}, \acute{E})_i \right) \Cup (\tilde{\Upsilon}, \acute{E})^c.$$

Since $(\tilde{\Upsilon}, \acute{E})^c$ is QPNS p-open (because $(\tilde{\Upsilon}, \acute{E})$ is given to be QPNS p-closed), the family

$$\{(\tilde{\mathcal{J}}, \acute{E})_i : i \in I\} \Cup \{(\tilde{\Upsilon}, \acute{E})^c\}$$

forms a QPNS p-open cover of $\tilde{\mathcal{M}}$.

Given that $\tilde{\mathcal{M}}$ is QPNS p-compact, this cover must be reducible to a finite subcover. That is, there exists a finite subcollection

$$\{(\tilde{\mathcal{J}}, \acute{E})_{i_k} : k = 1, 2, \ldots, n\}$$

such that

$$\tilde{\mathcal{M}} = \bigcup_{k=1}^{n} (\tilde{\mathcal{J}}, \acute{E})_{i_k} \Cup (\tilde{\Upsilon}, \acute{E})^c.$$

Taking the intersection with $(\tilde{\Upsilon}, \acute{E})$ on both sides, we obtain:

$$(\tilde{\Upsilon}, \acute{E}) = \left( \bigcup_{k=1}^{n} (\tilde{\mathcal{J}}, \acute{E})_{i_k} \Cup (\tilde{\Upsilon}, \acute{E})^c \right) \Cap (\tilde{\Upsilon}, \acute{E}).$$

Since $(\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{\Upsilon}, \acute{E})^c = \tilde{\Phi}$, it follows that:

$$(\tilde{\Upsilon}, \acute{E}) \subseteq \bigcup_{k=1}^{n} (\tilde{\mathcal{J}}, \acute{E})_{i_k}.$$

Thus, the QPNS p-open cover $\{(\tilde{\mathcal{J}}, \acute{E})_i : i \in I\}$ of $(\tilde{\Upsilon}, \acute{E})$ is reducible to a finite QPNS subcover. Hence, $(\tilde{\Upsilon}, \acute{E})$ is QPNS p-compact.

**Theorem 13.** *Let $(\tilde{\Upsilon}, \acute{E})$ be a QPNS p-compact subset of a QPNS p-Hausdorff space $\tilde{\mathcal{M}}$, and let $s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle} \in \tilde{\mathcal{M}} - (\tilde{\Upsilon}, \acute{E})$. Then there exist QPNS p-open sets $(\tilde{\mathcal{J}}, \acute{E})$ and $(\tilde{\mathcal{L}}, \acute{E})$ such that*

$$s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle} \in (\tilde{\mathcal{J}}, \acute{E}), \quad (\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\mathcal{L}}, \acute{E}), \quad \text{and} \quad (\tilde{\mathcal{J}}, \acute{E}) \Cap (\tilde{\mathcal{L}}, \acute{E}) \cong \tilde{\Phi}.$$

*Proof.* Since $\tilde{\mathcal{M}}$ is a QPNS p-Hausdorff space, for each $s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle} \in (\tilde{\Upsilon}, \acute{E})$, there exist QPNS p-open sets $(\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle}}, \acute{E})$ and $(\tilde{\mathcal{L}}_{s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle}}, \acute{E})$ such that

$$(\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle}}, \acute{E}) \Cap (\tilde{\mathcal{L}}_{s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle}}, \acute{E}) \cong \tilde{\Phi}.$$

Now, $\{(\tilde{\mathcal{L}}_x, \acute{E}) : s^{\curlywedge}_{\langle p_1, p_2, p_3, p_4 \rangle} \in (\tilde{\Upsilon}, \acute{E})\}$ is a QPNS p-open cover of $(\tilde{\Upsilon}, \acute{E})$. Since $(\tilde{\Upsilon}, \acute{E})$ is QPNS p-compact, there exist finitely many points

$$s^{\curlywedge}_{1, \langle p_1, p_2, p_3, p_4 \rangle}, s^{\curlywedge}_{2, \langle p_1, p_2, p_3, p_4 \rangle}, \ldots, s^{\curlywedge}_{n, \langle p_1, p_2, p_3, p_4 \rangle}$$

of $(\tilde{\Upsilon}, \acute{E})$ such that

$$(\tilde{\Upsilon}, \acute{E}) \subseteq \bigcup_{i=1}^{n} (\tilde{\mathcal{L}}_{s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}).$$

Define

$$(\tilde{\mathcal{L}}, \acute{E}) \cong \bigcup_{i=1}^{n} (\tilde{\mathcal{L}}_{s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}), \quad (\tilde{\mathcal{J}}, \acute{E}) \cong \bigcap_{i=1}^{n} (\tilde{\mathcal{J}}_{s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}).$$

Then $(\tilde{\mathcal{L}}, \acute{E})$ and $(\tilde{\mathcal{J}}, \acute{E})$ are clearly QPNS p-open sets such that

$$s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\mathcal{J}}, \acute{E}) \subseteq (\tilde{\mathcal{L}}, \acute{E}).$$

Moreover,

$$(\tilde{\mathcal{J}}, \acute{E}) \Cap (\tilde{\mathcal{L}}, \acute{E}) \cong \tilde{\Phi}.$$

If $y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\mathcal{L}}, \acute{E})$, then

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\mathcal{L}}_{s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E})$$

for some $s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}$, implying that

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin (\tilde{\mathcal{J}}_{s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}).$$

Since

$$(\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}) \Cap (\tilde{\mathcal{L}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}) \cong \tilde{\Phi},$$

it follows that

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin \bigcap_{i=1}^{n} (\tilde{\mathcal{J}}_{s^{\curlywedge}_{i,\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}).$$

Thus,

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin (\tilde{\mathcal{J}}, \acute{E}).$$

**Theorem 14.** *Let $(\tilde{\Upsilon}, \acute{E})$ and $(\tilde{\omega}, \acute{E})$ be disjoint QPNS p-compact subsets of a QPNS p-Hausdorff space $\tilde{\mathcal{M}}$. Then, there exist disjoint QPNS p-open sets $(\tilde{\mathcal{J}}, \acute{E})$ and $(\tilde{\mathcal{L}}, \acute{E})$ such that $(\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\mathcal{J}}, \acute{E})$ and $(\tilde{\omega}, \acute{E}) \subseteq (\tilde{\mathcal{L}}, \acute{E})$.*

*Proof.* Since $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\Upsilon}, \acute{E})$ and $(\tilde{\Upsilon}, \acute{E}) \Cap (\tilde{\mathcal{J}}, \acute{E}) \approx \tilde{\Phi}$, we have $y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin (\tilde{\mathcal{J}}, \acute{E})$.

Now, since $(\tilde{\omega}, \acute{E})$ is a QPNS p-compact subset of the QPNS p-Hausdorff space $\tilde{\mathcal{M}}$, and $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin (\tilde{\omega}, \acute{E})$, there exist QPNS p-open sets $(\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E})$ and $(\tilde{\mathcal{L}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E})$ such that $s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E})$, $(\tilde{\mathcal{J}}, \acute{E}) \subseteq (\tilde{\mathcal{L}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E})$, and $(\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}) \Cap (\tilde{\mathcal{L}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}) \approx \tilde{\Phi}$.

Clearly, $\{(\tilde{\mathcal{J}}_{s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}) : s^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\Upsilon}, \acute{E})\}$ is a QPNS p-open covering of $(\tilde{\Upsilon}, \acute{E})$. Since $(\tilde{\Upsilon}, \acute{E})$ is QPNS p-compact, there exist finitely many points $s_1{}^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}, s_2{}^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}, \ldots, s_n{}^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}$ in $(\tilde{\Upsilon}, \acute{E})$ such that

$$(\tilde{\Upsilon}, \acute{E}) \subseteq \bigcup_{i=1}^{n} (\tilde{\mathcal{J}}_{s_i{}^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}).$$

Let

$$(\tilde{\mathcal{J}}, \acute{E}) = \bigcup_{i=1}^{n} (\tilde{\mathcal{J}}_{s_i{}^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle}}, \acute{E}),$$

$$(\tilde{\mathcal{L}}, \acute{E}) = \bigcap_{i=1}^{n} (\tilde{\mathcal{L}}_{s_{i_{\langle p_1,p_2,p_3,p_4\rangle}^{\curlywedge}}}, \acute{E}).$$

Then, we have

$$(\tilde{\Upsilon}, \acute{E}) \subseteq (\tilde{\mathcal{J}}, \acute{E}) \quad \text{and} \quad (\tilde{\omega}, \acute{E}) \subseteq (\tilde{\mathcal{L}}, \acute{E}).$$

Since $(\tilde{\mathcal{J}}, \acute{E}) \subseteq (\tilde{\mathcal{L}}_{s_{i_{\langle p_1,p_2,p_3,p_4\rangle}^{\curlywedge}}}, \acute{E})$ for each $i$, it follows that $(\tilde{\mathcal{J}}, \acute{E})$ and $(\tilde{\mathcal{L}}, \acute{E})$ are QPNS p-open sets.

Also, $(\tilde{\mathcal{J}}, \acute{E}) \Cap (\tilde{\mathcal{L}}, \acute{E}) \approx \tilde{\Phi}$, because if

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\mathcal{J}}, \acute{E}),$$

then

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \in (\tilde{\mathcal{J}}_{s_{i_{\langle p_1,p_2,p_3,p_4\rangle}^{\curlywedge}}}, \acute{E})$$

for some $s_{i_{\langle p_1,p_2,p_3,p_4\rangle}^{\curlywedge}}$, which implies

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin (\tilde{\mathcal{L}}_{s_{i_{\langle p_1,p_2,p_3,p_4\rangle}^{\curlywedge}}}, \acute{E}).$$

Thus,

$$y^{\curlywedge}_{\langle p_1,p_2,p_3,p_4\rangle} \notin \bigcap_{i=1}^{n} (\tilde{\mathcal{L}}_{s_{i_{\langle p_1,p_2,p_3,p_4\rangle}^{\curlywedge}}}, \acute{E}) = (\tilde{\mathcal{L}}, \acute{E}).$$

Hence, the result follows.

**Theorem 15.** *Let $(\tilde{Y}, \tau_1, \tau_2, \acute{E})$ be a QPNS sub-space of $(\tilde{\mathcal{M}}, \tau_3, \tau_4, \acute{E})$. Then $\tilde{Y}$ is QPNS p-compact with respect to the QPNSBTS $\tau_1 \Cup \tau_2$ if and only if $\tilde{Y}$ is QPNS p-compact with respect to the QPNSBTS $\tau_3 \Cup \tau_4$.*

*Proof.* To prove that $\tilde{Y}$ is QPNS p-compact with respect to the QPNSTS $\tau_1 \Cup \tau_2$, let $\{(\tilde{\mathcal{L}}, \acute{E})_i : i \in I\}$ be a soft $\tau_1 \Cup \tau_2$ QPNS p-open cover of $\tilde{Y}$, then

$$\tilde{Y} \subseteq \bigcup_{i \in I} (\tilde{\mathcal{L}}, \acute{E})_i.$$

Since $(\tilde{\mathcal{L}}, \acute{E})_i \in \tau_1 \Cup \tau_2$, there exists $(\tilde{\mathcal{J}}, \acute{E})_i \in \tau_3 \Cup \tau_4$ such that

$$(\tilde{\mathcal{L}}, \acute{E})_i = (\tilde{\mathcal{J}}, \acute{E})_i \Cap (\tau_3 \Cup \tau_4),$$

which implies that $(\tilde{\mathcal{L}}, \acute{E})_i \subseteq (\tilde{\mathcal{J}}, \acute{E})_i$. Thus,

$$\bigcup_{i \in I} (\tilde{\mathcal{L}}, \acute{E})_i \subseteq \bigcup_{i \in I} (\tilde{\mathcal{J}}, \acute{E})_i.$$

Since $\tilde{Y} \subseteq \bigcup_{i \in I} (\tilde{\mathcal{L}}, \acute{E})_i$, it follows that $\tilde{Y} \subseteq \bigcup_{i \in I} (\tilde{\mathcal{J}}, \acute{E})_i$.

As $\tilde{Y}$ is QPNS p-compact with respect to $\tau_3 \Cup \tau_4$, the cover $\{(\tilde{\mathcal{J}}, \acute{E})_i : i \in I\}$ has a finite subcover $\{(\tilde{\mathcal{J}}, \acute{E})_{i_r} : r = 1, 2, \ldots, n\}$. Thus,

$$\tilde{Y} \subseteq \bigcup_{r=1}^{n} (\tilde{\mathcal{J}}, \acute{E})_{i_r},$$

which implies that $\{(\tilde{\mathcal{L}}, \acute{E})_{i_r} : 1 \leq r \leq n\}$ is a soft $\tau_1 \Cup \tau_2$ QPNS p-open cover of $\tilde{Y}$, proving QPNS p-compactness.

Conversely, suppose $\tilde{Y}$ is QPNS p-compact with respect to $\tau_1 \Cup \tau_2$. Let $\{(\tilde{\mathcal{J}}, \acute{E})_i : i \in I\}$ be a soft $\tau_3 \Cup \tau_4$ QPNS p-open cover of $\tilde{Y}$. Then,

$$\tilde{Y} \subseteq \bigcup_{i \in I} (\tilde{\mathcal{J}}, \acute{E})_i.$$

Defining $(\tilde{\mathcal{L}}, \acute{E})_i = \tilde{Y} \Cap (\tilde{\mathcal{J}}, \acute{E})_i$, we get that $\{(\tilde{\mathcal{L}}, \acute{E})_i : i \in I\}$ is a $\tau_1 \Cup \tau_2$ QPNS p-open cover of $\tilde{Y}$, which must have a finite subcover $\{(\tilde{\mathcal{L}}, \acute{E})_{i_r} : 1 \leq r \leq n\}$. Thus,

$$\tilde{Y} \subseteq \bigcup_{r=1}^{n} (\tilde{\mathcal{L}}, \acute{E})_{i_r}.$$

Since $(\tilde{\mathcal{L}}, \acute{E})_{i_r} \subseteq (\tilde{\mathcal{J}}, \acute{E})_{i_r}$, it follows that

$$\tilde{Y} \subseteq \bigcup_{r=1}^{n} (\tilde{\mathcal{J}}, \acute{E})_{i_r},$$

which proves that $\tilde{Y}$ is QPNS p-compact with respect to $\tau_3 \Cup \tau_4$.

## 6. Comparative Analysis

The following Table 1 provides a detailed comparative analysis of the proposed methods, contrasting them with the established techniques discussed in [10]. This comparison highlights the strengths and weaknesses of each approach, offering insights into how the proposed methods perform relative to the established techniques across various key factors:

| **Feature/Aspect** | **Published Work (NSS & NSTS) [10]** | **Proposed Work (QPNSS & QPNSTS)** |
|---|---|---|
| **Basic Framework** | Neutrosophic set theory extends fuzzy sets (FS) and intuitionistic fuzzy sets (IFS) by adding indeterminacy. | Enhances neutrosophic set theory by introducing a more detailed division of indeterminacy into two components: relative truth (RT) and relative falsehood (RF). |
| **Indeterminacy Component** | Traditional neutrosophic set includes a single indeterminate value (neutral membership). | Divides indeterminacy into relative truth (RT) and relative falsehood (RF) to refine the concept of indeterminacy. |
| **Membership Attributes** | Standard neutrosophic sets have three membership functions: truth, indeterminacy, and falsehood. | The proposed QPNSS introduces four membership functions: absolute truth, relative truth, relative falsehood, and absolute falsehood. |
| **New Operations** | Operations like intersection, union, complement, and subset are defined, but with three components (truth, indeterminacy, falsehood). | New operations defined for QPNSS, including quadri-partitioned soft set, subsets, complement, set difference, null set, AND, and OR operations. |
| **Quadri-Partitioned Neutrosophic Soft Set** | No equivalent concept in standard neutrosophic set theory. | The main novelty: QPNSS, which partitions the indeterminate value into RT and RF, providing greater clarity in uncertain situations. |

| **Feature/Aspect** | **Published Work (NSS & NSTS) [10]** | **Proposed Work (QPNSS & QPNSTS)** |
|---|---|---|
| **Topological Space** | Neutrosophic topological spaces (NST) exist, but they deal with three membership functions (truth, indeterminacy, falsehood). | A new concept, QPNSTS, is defined, adding topological properties like preopen sets (p-open), interior, closure, compactness, and reducibility to finite subcovers. |
| **Compactness and Reducibility** | Compactness in neutrosophic topological spaces is studied, but it focuses on three membership functions. | Explores QPNS compactness, intersection of QPNS p-closed sets, and QPNS p-compact spaces, introducing new compactness-related concepts. |
| **Applications and Use Cases** | Applied in areas like decision-making, multi-criteria decision analysis, and uncertainty modeling. | Focus on providing clearer and more precise representations of uncertainty, with potential applications in soft topological spaces and complex decision-making. |
| **Theoretical Contribution** | Introduces a formal framework for handling uncertain, imprecise, or indeterminate information. | Advances neutrosophic theory by providing a more granular structure (quadri-partitioned sets) and extending it to topological spaces with additional properties. |
| **Key Introduced Concept** | Neutrosophic sets are based on a single indeterminate value for information representation. | The introduction of QPNSS (quadri-partitioned neutrosophic soft set) and QPNSTS (quadri-partitioned neutrosophic soft topological space) enhances the theoretical framework for understanding uncertainty. |

Table 1: Comparison of Published Work (NSS and NSTS) and Proposed Work (QPNSS and QPNSTS)

## 7. Applications

Here are six potential applications of neutrosophic set theory, particularly the concept of the quadri-portioned neutrosophic soft set (QPNSS):

(i) **Decision-Making in Uncertain Environments:** Neutrosophic set theory can be applied in decision-making processes where there is ambiguity or incomplete information. The QPNSS framework allows for more accurate decisions by considering both relative truth and relative falsehood, offering a better representation of uncertainty in fields like engineering, economics, and management.

(ii) **Fault Diagnosis and Error Detection:** In systems where error detection is essential, such as in machine learning or fault diagnosis of complex systems, the QPNSS framework provides a more refined model by dividing indeterminacy into components of relative truth and relative falsehood. This allows for improved detection and error reduction in systems with uncertain or ambiguous data.

(iii) **Medical Diagnosis and Healthcare:** Neutrosophic set theory can be applied to medical diagnosis, where there is often incomplete or ambiguous information about a patient's condition. The use of QPNSS allows for better handling of indeterminate symptoms, improving diagnostic accuracy by incorporating both positive and negative evidence, enhancing the healthcare decision-making process.

(iv) **Image and Signal Processing:** The QPNSS framework can be applied in image and signal processing where there may be noise or uncertainty in the data. By using the four membership values (absolute truth, relative truth, relative falsehood, and absolute falsehood), the QPNSS model can help enhance image quality, edge detection, or signal filtering, leading to more precise and effective data interpretation.

(v) **Multi-Criteria Decision Analysis (MCDA):** In situations involving multiple conflicting criteria, such as in business, policy-making, or resource allocation, neutrosophic set theory can be used to model the uncertainty of criteria weights and evaluations. The QPNSS allows for a more nuanced understanding of these conflicts, improving the decision-making process by distinguishing between truth, falsehood, and indeterminacy in each criterion.

(vi) **Knowledge Representation and Reasoning in AI:** In artificial intelligence (AI), especially in the domain of knowledge representation and reasoning, neutrosophic set theory can enhance the ability to represent and reason with incomplete, uncertain, or contradictory knowledge. The QPNSS model helps AI systems to better handle cases where information is uncertain or partially known, allowing for more robust reasoning in uncertain environments, such as robotics or expert systems.

## 8. Conclusion and Future Work

Neutrosophic set theory is an important mathematical framework that is recognized as superior to existing theories of error reduction. This theory extends fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs). Its effectiveness can be enhanced by improving the definition of indeterminacy a mathematical term for situations in which values cannot be precisely determined. In this paper, methods for improving sensitivity and accuracy with respect to indeterminacy are presented.

According to the proposed approach, the indeterminate value is divided into two parts based on membership: relative truth (RT), which represents indeterminacy leaning towards truth, and relative falsehood (RF), which represents indeterminacy leaning towards falsehood. Indeterminacy is considered RT when it leans increasingly towards truth without being classified as true. In other words, this component reflects the extent to which the uncertain value tends to be accurate. It represents situations in which there is a greater likelihood of truth indicated by the context or evidence, but the value cannot be definitively determined to be true. Conversely, it is referred to as RF when it leans more towards untruth without being clearly false. This component reflects the extent to which the uncertain value tends to be false. Similarly, it represents situations where the evidence points to a greater probability of falsity, but the value cannot be definitively labeled as such.

Uncertainty arises when there is only one indeterminate value, as it remains unclear whether it favors true or false membership. The findings are more accurate when both RT and RF are captured compared to using a single indeterminate value. This differentiation enhances the overall accuracy of the uncertain scenario. The modified neutrosophic set is referred to as a quadri-portioned neutrosophic soft set (QPNSS). This model contains four membership attributes that are extremely relevant in real-world situations: absolute truth, relative truth, relative false, and absolute false, allowing for greater clarity in uncertain situations.

Extremely new operations are defined on QPNSS, including quadri-partitioned neutrosophic soft set, quadri-partitioned neutrosophic soft sub-sets, complement of quadri-partitioned neutrosophic soft set, absolute quadri-partitioned neutrosophic soft set, quadri-partitioned neutrosophic soft difference of sets, and absolute null quadri-partitioned neutrosophic soft set. In addition to this, AND and OR operations are also defined.

Quadri-partitioned neutrosophic soft topological space (QPNSTS) is defined, and the basic results related to this structure are addressed. Three new definitions are given, and based on one of these, known as pre-open (p-open) sets, some results are established. Interior, closure, and related results using these concepts are also addressed. Examples are developed for a clear understanding.

In the context of QPNSBTS and QPNS p-compact spaces, the idea of QPNS compactness is examined in this section. The concept of reducibility to finite sub-covers is one of the features of a QPNS cover that we define and study. Important theorems prove that the intersection of QPNS p-closed sets is non-empty and that a QPNS p-compact space preserves the finite intersection property among its QPNS p-closed sets. We also prove the existence of disjoint QPNS p-open sets separating disjoint QPNS p-compact subsets in a QPNS p-Hausdorff space and show that a QPNS p-closed subset of a QPNS p-compact space remains QPNS p-compact. The theoretical underpinning of QPNS spaces is improved by this work, which advances our knowledge of compactness in soft topological spaces.

In the future, we will examine the real-world applications of QPNSS in domains such as artificial intelligence, risk assessment, and decision-making. We will create algorithms for pattern identification and data analysis based on QPNSS. These algorithms may perform better in complex data sets by taking advantage of the increased sensitivity to indeterminacy. We will explore machine learning techniques to examine how QPNSS can be integrated into machine learning models, especially when dealing with ambiguous or insufficient data.

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## Declarations

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## Nomenclature

The following abbreviations are used in this manuscript:

- CST: Crisp set theory
- FST: Fuzzy set theory
- IFST: Intuitionistic fuzzy set theory
- NS: Neutrosophic set
- SVNS: Single valued neutrosophic set
- NST: Neutrosophic set theory
- SST: Soft set theory

- FSST: Fuzzy soft set theory
- VSST: Vague soft set theory
- NSST: Neutrosophic soft set theory
- DVNS: Double-valued neutrosophic set
- DVNI: Double-valued neutrosophic information
- DRINW: Double refined indeterminacy neutrosophic weighted
- TRINS: Triple refined indeterminate neutrosophic set
- SNHNS: Single valued heptapartitioned neutrosophic set.

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