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Fixed Points for Generalized Contractions in b-Gauge Spaces and Applications

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Abstract. In this work, we extend and generalize, the concept of α-Ψ contraction mappings in the setting of b-gauge spaces, where a new aspect of extension has been added . Subsequently, we give some related fixed point results that generalize many existing ones in the literature on this topic. Some of their applications to nonlinear integral equations on unbounded domains, including fractional differential equations with maxima, are also presented.

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1. Introduction

Banach's contraction principle, is one of the most important and significant results in the fixed point theory. Due to its effective applications in various areas of pure and applied mathematics, it has attracted a wide research interest in this theory. Indeed, the related existing literature is fulled with different results extending Banach's principle in two main directions: in the sense of the contraction mappings or (and) in the frame of generalized spaces. The metric space has been generalized in many different directions. One of the most main generalizations directly related to this work, is the gauge space

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which come back to Dugundji [1]. Briefly, a gauge space is a topological space whose topology is generated by a separating family of pseudo-metrics. It is distinguished from the metric space by the fact that the distance between two distinct points may be zero. For more details on the gauge spaces and related fixed point results, we refer to [1–5]. An other generalization of the metric space which relaxes the triangular inequality's axiom, is known in the fixed point theory as b-metric space. To be more precise, we state the following definition.

Definition 1.1. [6] Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $d : X \times X \longrightarrow \mathbb{R}_+$ is said to be a b-metric, if for all $u, v, w \in X$, the following conditions hold true

(b1) $d(u, v) = 0$ if and only if $u = v$;

(b2) $d(u, v) = d(v, u)$;

(b3) $d(u, w) \leq s\left[d(u, v) + d(v, w)\right]$.

In this case, (X, d) is called a b-metric space with constant s.

It should be noted that this structure is found in the literature under other names such as quasi-metric space [7] and metric type space [8]. For more information on the concept and origins of b-metric spaces, we reefer to the recent survey [9]. Recently, Ali et al. [10] extended gauge spaces in the setting of b-pseudo metrics and introduced the so called b-gauge spaces and proved some fixed point results for multi-valued mappings in this new space. Further generalizations of the metric structure, such as generalized metric space (known as Branciari metric space), rectangular b-metric space (known as Branciari b-metric space) and extended b-metric space and other generalized metric spaces can be found in [11–26].

The following generalized contraction condition called an α-ψ contraction in a metric space (X, d) is introduced and fixed point results for such type of contractions are established by Samet et al. [27]

$$\alpha(x, y)d(Fx, Fy) \leq \psi(d(x, y)), \ \ \forall x, y \in X,$$

where α and ψ are auxiliary functions satisfying some conditions. Many other results in this direction have been obtained later in the setting of b-metric spaces and gauge spaces with applications, see e.g. [5, 28–35] and the references therein. While so far in the existing literature, there are not enough contributions on this or even other trends in the frame of b-gauge spaces, expect in a few papers such as [36–38].

Motivated by the last observation and inspired by [27, 33, 39], we aim through this work to extend and generalize the concept of α-Ψ contraction mappings in the setting of b-gauge spaces, where a new aspect of extension has been added. Subsequently, we give some related fixed point results that generalize many existing ones in the literature on this topic. Some of their applications to nonlinear integral equations on unbounded domains, including fractional differential equations with maxima, are also presented.

2. Preliminaries

We start by recollecting some definitions from [10] to define b-gauge spaces introduced therein.

Definition 2.1. [10]

Let $\mathbf{E}$ be a non-empty set and let $s \geq 1$ be a given real number. A mapping $d : \mathbf{E} \times \mathbf{E} \longrightarrow \mathbb{R}_+$ is said to be a b-pseudo metric on $\mathbf{E}$, if for all $u, v, w \in \mathbf{E}$, the following conditions hold true

1. $d(u, u) = 0$;
2. $d(u, v) = d(v, u)$;
3. $d(u, w) \leq s\,[d(u, v) + d(v, w)]$.

The d-ball of radius $\epsilon > 0$ centred at $u \in \mathbf{E}$ is the set:

$$B(u, d, \epsilon) = \{v \in \mathbf{E} :\ d(u, v) < \epsilon\}.$$

Definition 2.2. [10] A family $\mathfrak{D} = \{d_\nu\}_{\nu \in \mathcal{N}}$ of b-pseudo metrics on $\mathbf{E}$ is said to be separating if for every two distinct points u and v, there exists $d_\nu \in \mathfrak{D}$ such that $d_\nu(u, v) \neq 0$.

Definition 2.3. [10] Let $\mathbf{E}$ be a nonempty set and $\mathfrak{D} = \{d_\nu\}_{\nu \in \mathcal{N}}$ a family of b-pseudo metrics on $\mathbf{E}$. The topology generated by the family $\mathfrak{D}$ and denoted by $\mathcal{T}(\mathfrak{D})$, is the topology whose subbase $\mathcal{B}(\mathcal{T})$ is the family of all balls $d_\nu(u, \epsilon)$, namely:

$$\mathcal{B}(\mathcal{T}) = \{d_\nu(u, \epsilon) :\ u \in \mathbf{E},\ \epsilon > 0,\ \nu \in \mathcal{N}\}.$$

The pair $(\mathbf{E}, \mathcal{B}(\mathcal{T}))$ is called a b-gauge space and is Hausdorff if $\mathfrak{D}$ is separating.

The notions of convergent sequences, Cauchy sequences and completeness in b-gauge spaces, are similar to those in metric spaces. For more details on these notions and further properties and examples on b-gauge spaces, we refer to [10].

In the aim of generalizing the contraction conditions, various families of auxiliary functions are introduced in the existing literature. In this regard, we introduce now one of such families.

For $s \geq 1$, let $\mathbf{\Psi^s}$ be the family of functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying the following conditions, where ψ^i denotes the i^{th} iteration of ψ.

$(\mathbf{\Psi^s}_1) :$ ψ is non-decreasing;

$(\mathbf{\Psi^s}_2) :$ $\psi(st) = s\psi(t),\ \ \forall\, t > 0$;

$(\mathbf{\Psi^s}_3) :$ $\sum\limits_{i=1}^{\infty} s^i \psi^i(t) < +\infty$ for each $t > 0$;

$(\mathbf{\Psi^s}_4) :$ $\psi(t_1) + \psi(t_2) \leq \psi(t_1 + t_2),\ \ \forall\, t_1, t_2 > 0$.

Example 1.

(i) Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the function defined by: $\psi(t) = c\,t$.
Then, $\psi \in \mathbf{\Psi^s}$, for all $s \geq 1$ such that $sc < 1$.

(ii) Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the function defined by: $\psi(t) = \begin{cases} \frac{t^2}{2} : 0 \leq t < 1 \\ \frac{t}{2} : t \geq 1 \end{cases}$
Then $\psi \in \mathbf{\Psi^1}$.

Lemma 2.4.
For every $\psi \in \mathbf{\Psi^s}$, the following properties are satisfied:

(i) $\psi(t) \leq \psi(st) < t, \ \forall t > 0$;

(ii) $\lim\limits_{t \to 0^+} \psi(t) = 0$.

Proof.
We begin by demonstrating the following statement:

$$\psi(st) < t, \ \forall t > 0. \tag{2.1}$$

To this end, we proceed by contradiction.
Let us suppose that $\psi(st_0) \geq t_0$ for some $t_0 > 0$. From $(\mathbf{\Psi^s}_1)$ and $(\mathbf{\Psi^s}_2)$, we get:

$$s^2\psi^2(t_0) = s\psi(\psi(st_0)) \geq s\psi(t_0) = \psi(st_0) \geq t_0.$$

Similarly it can be easily deduced by induction that:

$$\forall i \geq 1 : \ \ s^i\psi^i(t_0) \geq t_0.$$

Consequently:

$$\lim_{i \to \infty} s^i\psi^i(t_0) \geq t_0 > 0,$$

which is a contradiction with $(\mathbf{\Psi^s}_3)$. Hence, (2.1) is proved. The first inequality in the statement (1) follows directly from $(\mathbf{\Psi^s}_1)$ (recall that $s \geq 1$).
Note that from (1), we have:

$$0 \leq \lim_{t \to 0^+} \psi(t) \leq \lim_{t \to 0^+} t = 0.$$

Hence, (2) is proved.

Remark 2.5.
Note that if ψ is a function satisfying $(\mathbf{\Psi^s}_1)$, $(\mathbf{\Psi^s}_2)$ and $(\mathbf{\Psi^s}_4)$ such that $\psi(st) < t$, then to conclude that $\psi \in \mathbf{\Psi}^s$, it is sufficient to show that $\dfrac{\psi(s\,.)}{.}$ is non-decreasing. Indeed, we have:

$$\frac{s^{i+1}\psi^{i+1}(t)}{s^i\psi^i(t)} = \frac{s\psi^{i+1}(t)}{\psi^i(t)} = \frac{\psi\left(s\psi^i(t)\right)}{\psi^i(t)}.$$

On the other hand, from the statement (1) in Lemma 2.4, we deduce by induction that:

$$\forall i \geq 1: \;\; \psi^i(t) < t.$$

Hence, from the fact that $\dfrac{\psi(s\,.)}{.}$ is non-decreasing, we obtain:

$$\frac{s^{i+1}\psi^{i+1}(t)}{s^i\psi^i(t)} \leq \frac{\psi(st)}{t} < \frac{t}{t} = 1,$$

which is a sufficient condition leading to $(\mathbf{\Psi^s}_3)$.

For a mapping $\alpha : \mathbf{E} \times \mathbf{E} \longrightarrow \mathbb{R}_+$ and a non-decreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, such that $\sum\limits_{i=1}^{\infty} \psi^i(t) < +\infty$ for all $t > 0$, the concepts of α-admissible mappings and α-Ψ contraction mappings in a metric space $(\mathbf{E}, d)$, were introduced for the first time by Samet et al. [27].

Definition 2.6.

A map $F : \mathbf{E} \longrightarrow \mathbf{E}$ is said to be

- α-admissible, if for all $x, y \in \mathbf{E}$: $\alpha(x, y) \geq 1$ implies $\alpha(Fx, Fy) \geq 1$
- α-Ψ contraction mapping, if

$$\alpha(x, y)d(Fx, Fy) \leq \psi(d(x, y)), \;\; \forall x, y \in \mathbf{E}.$$

Later, other contraction conditions of such type have been considered by many authors to extend the Banach's principle. In these results, the following condition for α-admissible mappings F, is often imposed

$$\exists\, x^0 \in \mathbf{E}, \;\text{ such that }\; \alpha(x^0, Fx^0) \geq 1. \tag{2.2}$$

In [33], the authors introduced the following relaxed condition

$$\exists N \in \mathbb{N}^*,\, \exists\, (x^p)_{p=0}^N \subset \mathbf{E}, \text{ with } x^N = Fx^0, \text{ such that } \alpha(x^{p-1}, x^p) \geq 1,\, \forall p = 1, .., N. \tag{2.3}$$

Where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For α-admissible mapping F, it is clear that if (2.2) is satisfied, then (2.3) is satisfied too with $N = 1$. But the following simple example illustrates that the converse is not necessarily true.

Example 2.

Let $X = \{0, 1, 2, 3\}$,

$$\begin{aligned} F:\;\; & X \longrightarrow X \\ & 0 \mapsto 1 \\ & 1 \mapsto 0 \\ & 2 \mapsto 3 \\ & 3 \mapsto 2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} & \alpha : X \times X \longrightarrow \{0, 1\} \\ & \alpha(x, y) = \begin{cases} 0 : \; (x, y) \in \{(0, 1), (1, 0), (2, 3), (3, 2)\} \\ 1 : \; \text{otherwise} \end{cases} \end{aligned}.$$

It can be easily seen that F is α-admissible and (2.2) is not satisfied.

Whereas, there exist $x^0 = 2$, $x^1 = 1$ and $x^2 = Fx^0 = 3$, such that $\alpha(x^0, x^1) = \alpha(x^1, x^2) = 1 \geq 1$. That is (2.3) is satisfied with $N = 2$.

3. Main results

Throughout the sequel, $\mathbf{E}$ is a non-empty set endowed with a separating complete b-gauge structure $\mathfrak{D} = \{d_\nu\}_{\nu\in\mathcal{N}}$, where $\mathcal{N}$ is an index set.

Inspired by [27, 39], we give in what follows generalized concepts of α-admissibility and α-Ψ contractivity in the setting of b-gauge spaces. To this end, we start by introducing the following auxiliary family and mapping.

We denote by $\boldsymbol{\alpha}_\nu$, the following family: $\boldsymbol{\alpha}_\nu = \{\alpha_\nu : \mathbf{E}\times\mathbf{E} \longrightarrow \mathbb{R}_+\}_{\nu\in\mathcal{N}}$.

$w : \mathcal{N} \longrightarrow \mathcal{N}$ is a mapping from the index set $\mathcal{N}$ into itself, such that:

$$\forall \nu \in \mathcal{N},\ \forall u, v \in \mathbf{E}:\ \ d_\nu(u, v) \le d_{w(\nu)}(u, v). \tag{3.1}$$

Definition 3.1.

A mapping $F : \mathbf{E} \longrightarrow \mathbf{E}$ is said to be $\boldsymbol{\alpha}_\nu$-admissible, if

$$\forall \nu \in \mathcal{N},\ \forall u, v \in \mathbf{E}:\ \ \alpha_\nu(u, v) \ge 1 \text{ implies } \ \alpha_\nu(Fu, Fv) \ge 1.$$

Definition 3.2.

Let $F : \mathbf{E} \longrightarrow \mathbf{E}$ be a given mapping and $\{\psi_\nu\}_{\nu\in\mathcal{N}} \subset \mathbf{\Psi}^s$.

F is said to be a generalized ($\boldsymbol{\alpha}_\nu$,$\mathbf{\Psi}^s$,w) contraction if

$$\alpha_\nu(u, v)\, d_\nu(Fu, Fv) \le \psi_\nu\left(d_{w(\nu)}(u, v)\right),\ \forall u, v \in \mathbf{E},\ \forall \nu \in \mathcal{N}. \tag{3.2}$$

Remark 3.3. It should be noted that many α-Ψ contractive type mappings in the literature are generalized by that given in (3.2) in two distinct aspects. The introduction of a family of mappings $\boldsymbol{\alpha}_\nu = \{\alpha_\nu\}_{\nu\in\mathcal{N}}$ instead of only one mapping α is the clear first aspect of generalization. While the introduction of the mapping w is the second one. Indeed, since some α-Ψ contraction conditions introduced in similar studies in this direction correspond to $w = I_{\mathcal{N}}$ [4, 10, 33, 36], then in view of (3.1), our contraction condition (3.2) is weaker than those mentioned above.

Example 3. Let X be the space of all real sequences:

$$X = \{u = (u_1, u_2, ..., u_n, ...) : u_n \in \mathbb{R},\ n \in \mathbb{N}^*\}.$$

For each $n \in \mathbb{N}^*$, let $\pi_n : X \longrightarrow \mathbb{R}$ be the mapping defined by $\pi_n(u) = u_n$.

Let $\{d_n\}_{n\in\mathbb{N}^*}$ be the family of b-pseudo-metrics with constant $s = 2$ defined on X by

$$d_n(u, v) = |\pi_n(u) - \pi_n(v)|^2$$

Let $w : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ be the mapping defined by $w(n) = n + 1$.

Consider the map $F : X \longrightarrow X$ defined as follows:

$$Fu = \begin{cases} ((1-\frac{1}{2})(2-u_2), (1-\frac{2}{3})(2-u_3), ..., (1-\frac{n}{n+1})(2-u_{n+1}), ...), & \exists n \in \mathbb{N}^* : u_n \le 2; \\ \\ (2u_2 - 2, 2u_3 - 2, ..., 2u_{n+1} - 2, ...), & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{\alpha}_n = \{\alpha\}$, where $\alpha : X \times X \longrightarrow \mathbb{R}_+$ is the function given by:

$$\alpha(u,v) = \begin{cases} 1 : \ u_n, v_n \leq 2 \ \text{ for some } \ n \in \mathbb{N}^* \\ 0 : \ \text{otherwise} \end{cases}$$

Now, let $\mathbf{\Psi^2}$ be the family of the functions ψ_n defined for each $n \in \mathbb{N}^*$ by:

$$\psi_n(t) = \frac{1}{(n+1)^2}\, t$$

Let $u, v \in X$, we distinguish two cases:

Case 1: There exists $n \in \mathbb{N}^*$ such that $u_n, v_n \leq 2$. Then:

$$\begin{aligned} \alpha(u,v) d_n(Fu,Fv) &= d_n(Fu,Fv) = \left|(1 - \tfrac{n}{n+1})(2 - u_{n+1}) - (1 - \tfrac{n}{n+1})(2 - v_{n+1})\right|^2 \\ &= (1 - \tfrac{n}{n+1})^2 \left|u_{n+1} - v_{n+1}\right|^2 = \tfrac{1}{(n+1)^2} \left|u_{n+1} - v_{n+1}\right|^2 \\ &= \psi_n\left(d_{n+1}(u,v)\right) = \psi_n\left(d_{w(n)}(u,v)\right) \end{aligned}$$

Case 2: For every $n \in \mathbb{N}^* : u_n > 2$ or $v_n > 2$. Since $\alpha(u,v) = 0$, clearly we have:

$$\alpha(u,v) d_n(Fu,Fv) = 0 \leq \psi_n\left(d_{w(n)}(u,v)\right)$$

Consequently, F is a generalized $(\boldsymbol{\alpha}_n, \mathbf{\Psi}^2, w)$ contraction.

We state now our first main result.

Theorem 1.

Let $F : \mathbf{E} \longrightarrow \mathbf{E}$ be a a generalized $(\boldsymbol{\alpha}_\nu, \mathbf{\Psi}^s, w)$ contraction. Suppose that the following conditions hold:

$(C1)$ *F is $\boldsymbol{\alpha}_\nu$-admissible.*

$(C2)$ $\exists\, x^0 \in \mathbf{E}$, $N \in \mathbb{N}^*$ *and* $(a_0^p)_{p=0}^N \subset \mathbf{E}$, *with* $a_0^0 = x^0$ *and* $a_0^N = Fx^0$, *such that:*

(i) $\alpha_\nu(a_0^{p-1}, a_0^p) \geq 1, \ \forall p = 1, .., N, \ \forall \nu \in \mathcal{N};$

(ii) $\displaystyle\sum_{p=1}^{N} s^p d_{w^i(\nu)}(a_0^{p-1}, a_0^p) \leq M_{s,\nu}(x^0) < +\infty, \ \forall i \in \mathbb{N}, \ \forall \nu \in \mathcal{N}.$

$(C3)$ $\forall \nu \in \mathcal{N}, \ \exists \tilde{\psi_\nu} \in \mathbf{\Psi}^s : \ \psi_{w^i(\nu)} \leq \tilde{\psi_\nu}, \ \forall i \in \mathbb{N}$

$(C4)$ *(i) F is continuous or*

(ii) For every sequence $\left\{u^k\right\}_{k\in\mathbb{N}}$ *of* $\mathbf{E}$*, such that for all* $k\in\mathbb{N}$ *and the same positive integer* N *given in* $(C2)$*:*

$$\exists\,(a_k^p)_{p=0}^N\subset\mathbf{E},\ s.t.\ a_k^0=u^k,\ a_k^N=u^{k+1}\ and\ \alpha_\nu(a_k^{p-1},a_k^p)\geq 1,\ \forall p=\overline{1,N},\forall\nu\in\mathcal{N}, \tag{3.3}$$

if $u^k\xrightarrow[k\to\infty]{}u$*, then there exists a sub-sequence* $\left\{u^{k_l}\right\}_{l\in\mathbb{N}}$ *of* $\left\{u^k\right\}_{k\in\mathbb{N}}$ *and* $l_0\in\mathbb{N}$ *such that* $\alpha_\nu(u^{k_l},u)\geq 1$ *for all* $l\geq l_0$*.*

Then, F *has a fixed point.*

Proof.
Note first that according to $(C2(i))$ and $(C1)$, we deduce by induction that

$$\forall p=1,...,N,\ \forall k\in\mathbb{N},\ \forall\nu\in\mathcal{N}:\ \alpha_\nu(F^k a_0^{p-1},F^k a_0^p)\geq 1.$$

Consequently, using (3.2), the following inequalities hold true:

$$\begin{aligned} d_\nu\left(F^k a_0^{p-1},F^k a_0^p\right) &\leq \alpha_\nu(F^{k-1}a_0^{p-1},F^{k-1}a_0^p)d_\nu\left(F^k a_0^{p-1},F^k a_0^p\right)\\ &\leq \psi_\nu\left(d_{w(\nu)}\left(F^{k-1}a_0^{p-1},F^{k-1}a_0^p\right)\right), \end{aligned}$$

for all $\nu\in\mathcal{N}$, $k\in\mathbb{N}$ and $p=1,...,N$.

Now, since ψ_ν is non-decreasing for each $\nu\in\mathcal{N}$, repeated application of the previous inequalities yield:

$$d_\nu(F^k a_0^{p-1},F^k a_0^p)\leq\psi_\nu\left(\psi_{w(\nu)}\left(...\psi_{w^{k-1}(\nu)}\left(d_{w^k(\nu)}(a_0^{p-1},a_0^p)\right)...\right)\right),$$

for all $k\in\mathbb{N}$, $\nu\in\mathcal{N}$ and every $p=1,...,N$.

Hence, by means of $(C3)$, we obtain:

$$d_\nu(F^k a_0^{p-1},F^k a_0^p)\leq\tilde{\psi_\nu}^k\left(d_{w^k(\nu)}(a_0^{p-1},a_0^p)\right). \tag{3.4}$$

Let now x^0 be the element introduced in $(C2(i))$ and let $\left\{x^k\right\}_{k\in\mathbb{N}}$ be the sequence in $\mathbf{E}$ defined by $x^{k+1}=Fx^k$.

Assume that $x^{k+1}\neq x^k$ for all $k\in\mathbb{N}$, since otherwise the result is clear. Recall that $a_0^0=x^0$ and $a_0^N=Fx^0$, then for all $k\in\mathbb{N}$, we have:

$$d_\nu(F^k x^0,F^{k+1}x^0)\leq sd_\nu(F^k a_0^0,F^k a_0^1)+s^2 d_\nu(F^k a_0^1,F^k a_0^2)+...+s^N d_\nu(F^k a_0^{N-1},F^k a_0^N).$$

Hence, using (3.4) together with $(\mathbf{\Psi^s}_1)$, $(\mathbf{\Psi^s}_2)$ and $(\mathbf{\Psi^s}_4)$, we obtain:

$$\begin{aligned}
d_\nu(F^k x^0, F^{k+1}x^0) &\leq s\tilde{\psi_\nu}^k\left(d_{w^k(\nu)}(a_0^0,a_0^1)\right) + s^2\,\tilde{\psi_\nu}^k\left(d_{w^k(\nu)}(a_0^1,a_0^2)\right) + ... + s^N\,\tilde{\psi_\nu}^k\left(d_{w^k(\nu)}(a_0^{N-1},a_0^N)\right)\\
&= \tilde{\psi_\nu}^k\left(s d_{w^k(\nu)}(a_0^0,a_0^1)\right) + \tilde{\psi_\nu}^k\left(s^2 d_{w^k(\nu)}(a_0^1,a_0^2)\right) + ... + \tilde{\psi_\nu}^k\left(s^N d_{w^k(\nu)}(a_0^{N-1},a_0^N)\right)\\
&= \tilde{\psi_\nu}\left(\tilde{\psi_\nu}^{k-1}\left(s d_{w^k(\nu)}(a_0^0,a_0^1)\right)\right) + \tilde{\psi_\nu}\left(\tilde{\psi_\nu}^{k-1}\left(s^2 d_{w^k(\nu)}(a_0^1,a_0^2)\right)\right) + ...\\
&+ \tilde{\psi_\nu}\left(\tilde{\psi_\nu}^{k-1}\left(s^N d_{w^k(\nu)}(a_0^{N-1},a_0^N)\right)\right)\\
&\leq \tilde{\psi_\nu}\left(\tilde{\psi_\nu}^{k-1}\left(s\, d_{w^k(\nu)}(a_0^0,a_0^1)\right) + \tilde{\psi_\nu}^{k-1}\left(s^2\, d_{w^k(\nu)}(a_0^1,a_0^2)\right) + ...\right.\\
&\left.+\tilde{\psi_\nu}^{k-1}\left(s^N\, d_{w^k(\nu)}(a_0^{N-1},a_0^N)\right)\right)\\
&= \tilde{\psi_\nu}\left(s\,\tilde{\psi_\nu}^{k-1}\left(d_{w^k(\nu)}(a_0^0,a_0^1)\right) + s^2\,\tilde{\psi_\nu}^{k-1}\left(d_{w^k(\nu)}(a_0^1,a_0^2)\right) + ...\right.\\
&\left.+s^N\,\tilde{\psi_\nu}^{k-1}\left(d_{w^k(\nu)}(a_0^{N-1},a_0^N)\right)\right).
\end{aligned}$$

Since ψ_ν is non-decreasing, repeated application of the above inequalities yields:

$$d_\nu(F^k x^0, F^{k+1}x^0) \leq \tilde{\psi_\nu}^k\left(\sum_{p=1}^{N} s^p\, d_{w^k(\nu)}(a_0^{p-1},a_0^p)\right).$$

Consequently, it follows from $(C2(ii))$:

$$d_\nu(F^k x^0, F^{k+1}x^0) \leq \tilde{\psi_\nu}^k\left(M_{s,\nu}\left(x^0\right)\right). \tag{3.5}$$

We are now ready to prove that $\left\{x^k\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Indeed, let $k\in\mathbb{N}$ and $m\in\mathbb{N}^*$. We have:

$$\begin{aligned}
d_\nu(F^k x^0, F^{k+m}x^0) &\leq s d_\nu\left(F^k x^0, F^{k+1}x^0\right) + s^2\, d_\nu\left(F^{k+1}x^0, F^{k+2}x^0\right) + ...\\
&+ s^{m-1}\, d_\nu\left(F^{k+m-2}x^0, F^{k+m-1}x^0\right) + s^m\, d_\nu\left(F^{k+m-1}x^0, F^{k+m}x^0\right).
\end{aligned}$$

It follows so from (3.5), that

$$\begin{aligned} d_\nu(F^k x^0, F^{k+m}x^0) &\leq s\tilde{\psi_\nu}^k\left(M_{s,\nu}(x^0)\right) + s^2\tilde{\psi_\nu}^{k+1}\left(M_{s,\nu}(x^0)\right) + ... + s^m\tilde{\psi_\nu}^{k+m-1}\left(M_{s,\nu}(x^0)\right) \\ &= \frac{1}{s^{k-1}}\left[s^k\tilde{\psi_\nu}^i\left(M_{s,\nu}(x^0)\right) + s^{k+1}\tilde{\psi_\nu}^i\left(M_{s,\nu}(x^0)\right) + ... + s^{k+m-1}\tilde{\psi_\nu}^i\left(M_{s,\nu}(x^0)\right)\right] \\ &\leq \frac{1}{s^{k-1}}\sum_{i=k}^{\infty} s^i\,\tilde{\psi_\nu}^i\left(M_{s,\nu}(x^0)\right). \end{aligned} \tag{3.6}$$

Since in view of $(\mathbf{\Psi^s}_3)$ together with the second statement of Lemma 2.4, we have:

$$\lim_{k\to\infty}\sum_{i=k}^{\infty} s^i\,\tilde{\psi_\nu}^i\left(M_{s,\nu}(x^0)\right) = 0,$$

then we deduce from (3.6) that $\left\{F^k x^0 = x^k\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the complete b-gauge space $\mathbf{E}$ and so convergent to some $x^* \in \mathbf{E}$. That is, for all $\nu \in \mathcal{N}$:

$$\lim_{k\to\infty} d_\nu(x^k, x^*) = 0.$$

On the other hand, the continuity of F guaranteed by $(C4(i))$, implies that for all $\nu \in \mathcal{N}$:

$$\lim_{k\to\infty} d_\nu(x^k, Fx^*) = d_\nu(Fx^{k-1}, Fx^*) = 0.$$

Thus, for all $\nu \in \mathcal{N}$:

$$d_\nu(x^*, Fx^*) \leq s\left(d_\nu(x^*, x^k) + d_\nu(x^k, Fx^*)\right) \xrightarrow[k\to\infty]{} 0.$$

Since the b-gauge structure is separating, we conclude that $x^* = Fx^*$.

Suppose now that $(C4(ii))$ is satisfied. Note first that $(C2(i))$ means that (3.3) is satisfied for $k = 0$. Since F is $\boldsymbol{\alpha}_\nu$-admissible, it follows by induction that (3.3) is satisfied for each $k \geq 1$ with $a_k^p = Fa_{k-1}^p$, for all $p = 0, ..., N$. Thus, according to $(C4(ii))$, there exists a sub-sequence $\left\{x^{k_l}\right\}_{l\in\mathbb{N}}$ and some l_0 such that for all $l \geq l_0$ and $\nu \in \mathcal{N}$, we have $\alpha_\nu(x^{k_l}, x^*) \geq 1$.

Hence, applying (3.2), we obtain:

$$d_\nu(Fx^{k_l}, Fx^*) \leq \alpha_\nu(x^{k_l}, x^*)\,d_\nu(Fx^{k_l}, Fx^*) \leq \psi_\nu\left(d_{w(\nu)}(x^{k_l}, x^*)\right).$$

Now, letting $l \longrightarrow \infty$ in the right hand side of the above inequality taking into account the second statement of Lemma 2.4, we deduce:

$$\lim_{l\to\infty} d_\nu(Fx^{k_l}, Fx^*) = 0, \ \ \forall \nu \in \mathcal{N}.$$

That is

$$\lim_{l\to\infty} Fx^{k_l} = Fx^*.$$

Noting that

$$\lim_{l\to\infty} x^{k_l+1} = \lim_{l\to\infty} Fx^{k_l},$$

we deduce that $x^* = Fx^*$. The proof is complete.

Remark 3.4. Condition $(C2(i))$ is an extension of (2.3) in the setting of b-gauge spaces. Thus, according to Example 2, this condition is weaker than the condition (2.2), frequently imposed in the existing literature on this topic, like in [37, 38, 40].

Sufficient conditions guaranteeing the uniqueness of the fixed point is given in the following theorem.

Theorem 2.
Let $F : \mathbf{E} \longrightarrow \mathbf{E}$ be a generalized $(\boldsymbol{\alpha}_\nu, \boldsymbol{\Psi}^s, w)$ contraction satisfying conditions $(C1)$, $(C3)$ and $(C4)$ in Theorem 1. Suppose that the following condition holds

$(\tilde{C2})$ $\forall x, y \in \mathbf{E}$ *with* $x \neq y$, *there exists* $N = N(x,y) \in \mathbb{N}^*$ *and* $(a^p_{x,y})^N_{p=0} \subset \mathbf{E}$ *such that:*

- *(i)* $a^0_{x,y} = x$, $a^N_{x,y} = y$, *and* $\alpha_\nu(a^{p-1}_{x,y}, a^p_{x,y}) \geq 1$, $\forall p = 1, ..., N, \forall \nu \in \mathcal{N}$;
- *(ii)* $\displaystyle\sum_{p=1}^{N} s^p d_{w^i(\nu)}(a^{p-1}_{x,y}, a^p_{x,y}) \leq M_{s,\nu}(x,y) < +\infty, \ \ \forall i \in \mathbb{N}, \ \forall \nu \in \mathcal{N}.$

Then F has a unique fixed point.

Proof.
The existence of a fixed point for F results from Theorem 1. Indeed, let x^0 be an arbitrary element in $\mathbf{E}$.

- If $x^0 = Fx^0$, then x^0 is a fixed point.
- If $x^0 \neq Fx^0$, then with $x = x^0$ and $y = Fx^0$ condition $(\tilde{C2})$ reduces to condition $(C2)$ in Theorem 1, from which follows that F has a fixed point.

Suppose now that x, y are two fixed points of F such that $x \neq y$. By means of $(C1)$, $(\tilde{C2})$, $(C3)$ and in a similar way as that used to get (3.5), we have also the following inequality:

$$d_\nu(F^k x, F^k y) \leq \tilde{\psi_\nu}^k \left(M_{s,\nu}(x,y)\right),$$

for all $k \in \mathbb{N}$ and all $\nu \in \mathcal{N}$. Hence

$$d_\nu(x,y) = d_\nu(F^k x, F^k y) \leq \tilde{\psi_\nu}^k \left(M_{s,\nu}(x,y)\right), \tag{3.7}$$

for all $k \in \mathbb{N}$ and all $\nu \in \mathcal{N}$. Noting that

$$\tilde{\psi_\nu}^k \left(M_{s,\nu}(x,y)\right) \leq s^k \, \tilde{\psi_\nu}^k \left(M_{s,\nu}(x,y)\right),$$

and by letting $k \longrightarrow \infty$ in (3.7) taking into account ($\mathbf{\Psi^s}_3$), we deduce:

$$d_\nu(x, y) = 0, \ \ \forall \nu \in \mathcal{N},$$

which is a contradiction with $x \neq y$, since $\mathfrak{D}$ is separating. The proof is complete.

Example 4. Let $X = \mathbb{R}$ be the complete b-gauge space with constant $s = 2$, endowed with the separated family of b-pseudo-metrics $\mathcal{D} = \{d_n, n \geq 1\}$ defined by: $d_n(x, y) = n(|x| - |y|)^2$. Let $Fx = \frac{x}{2}$, $\psi_n(t) = \frac{1}{n+1}t$ and $w(n) = (n + 1)^3$ for all $n \geq 1$.

$$\alpha_n(x, y) = \begin{cases} n, & x \neq y \\ 0, & \text{otherwise.} \end{cases}$$

For $x, y \in \mathbb{R}$ with $x \neq y$ we have:

$$\alpha_n(x, y)d_n(Fx, Fy) = \frac{n^2}{4}(|x| - |y|)^2$$

and

$$\psi_n(d_{(n+1)^3}(x, y)) = (n + 1)^2(|x| - |y|)^2$$

Then, for $x \neq y$ and $n \geq 1$ we have:

$$\alpha_n(x, y)d_n(Fx, Fy) \leq \psi_n(d_{w(n)}(x, y)) = (n + 1)^2(|x| - |y|)^2$$

Hence, F is a generalized ($\boldsymbol{\alpha}_n$,$\mathbf{\Psi}^2$,w) contraction. Let us now show that F verifies the other conditions of Theorem 2. Indeed, for (C_1) we have, for all $n \geq 1$

$$\alpha_n(x, y) \geq 1 \Rightarrow \alpha_n(Fx, Fy) = n \geq 1,$$

then, F is $\boldsymbol{\alpha}_n$-admissible.
It can be easily seen that ($C3$) is satisfied with $\tilde{\psi_n} = \psi_n$, for all $n \geq 1$.
Let $x, y \in \mathbb{R}$ such that $x \neq y$. Then, there exists $z \in \mathbb{R}$ such that $x \neq z$ and $y \neq z$. Hence, for all $n \geq 1$, we have $\alpha_n(x, z) = n \geq 1$ and $\alpha_n(x, z) = n \geq 1$. Then ($\tilde{C}2$)-($i$) holds with $N = 1$ and ($\tilde{C}2$)-(ii) follows immediately from the fact that $\psi_n^i \leq \psi_n$ for all $i \in \mathbb{N}$.
It is not hard to see that F is continuous and so (C_4) is satisfied. Thus, all conditions of Theorem 2 are fulfilled and consequently F has a unique fixed point, which is 0.

Let us state the following conditions

(P_{C2}) There exists $x^0 \in \mathbf{E}$ such that $\alpha_\nu(x^0, Fx^0) \geq 1$, $\forall \nu \in \mathcal{N}$ and furthermore: $d_{w^i(\nu)}(x^0, Fx^0) < +\infty$, $\forall i \in \mathbb{N}$, $\forall \nu \in \mathcal{N}$;

($\tilde{P}_{C2}$) $\forall x, y \in \mathbf{E}$ with $x \neq y$, there exists $z \in \mathbf{E}$ such that $\alpha_\nu(x, z) \geq 1$, and $\alpha_\nu(y, z) \geq 1$, $\forall \nu \in \mathcal{N}$;

(P_{C4}) (i) F is continuous, or

(ii) for every sequence $\left\{u^k\right\}_{k\in\mathbb{N}}$ of $\mathbf{E}$, such that $\alpha_\nu(u^{k-1}, u^k) \geq 1$, $\forall \nu \in \mathcal{N}$, if $u^k \xrightarrow[k\to\infty]{} u$, then there exists a sub-sequence $\left\{u^{k_l}\right\}_{l\in\mathbb{N}}$ of $\left\{u^k\right\}_{k\in\mathbb{N}}$ and $l_0 \in \mathbb{N}$ such that $\alpha_\nu(u^{k_l}, u) \geq 1$ for all $l \geq l_0$.

as spacial cases of $(C2)$, $(\tilde{C2})$ and $(C4)$ respectively. The following corollaries follow immediately from Theorem 1 and Theorem 2.

Corollary 3.5.

Let $F : \mathbf{E} \longrightarrow \mathbf{E}$ be a generalized ($\boldsymbol{\alpha}_\nu$,$\boldsymbol{\Psi}^s$,w) contraction. Suppose that in addition of conditions $(C1)$, $(C3)$ and $(C4)$ of Theorem 1, (P_{C2}) holds true. Then, F has a fixed point.

Corollary 3.6.

Let $F : \mathbf{E} \longrightarrow \mathbf{E}$ be a generalized ($\boldsymbol{\alpha}_\nu$,$\boldsymbol{\Psi}^s$,w) contraction. Suppose that in addition of conditions $(C1)$ and $(C3)$ of Theorem 1, conditions $(\tilde{P}_{C2})$ and (P_{C4}) hold. Then, F has a unique fixed point.

4. Application

In this section, we focus on the existence of solutions of some nonlinear integral equations as an application to the results proved in the previous section.

Let us consider the following integral equation:

$$x(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t G(t,\tau) f(\tau, x(\tau), gx(\tau)) d\tau, & t > 0 \\ \\ \varphi(t), & t \leq 0, \end{cases} \tag{4.1}$$

where $G : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, $f : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\varphi :]-\infty, 0] \longrightarrow \mathbb{R}$ are nonlinear continuous functions and $g : \mathcal{C}(\mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R})$ where $\mathcal{C}(\mathbb{R})$ denotes the set of all real continuous functions on $\mathbb{R}$ and gx is a delay function.

Let $\mathbf{E} = \mathcal{C}(\mathbb{R})$ be the complete b-gauge space with constant $s = 2$, endowed with the separated family of b-pseudo-metrics $\{d_K\}_{K\in\mathcal{K}}$ defined by:

$$d_K(x, y) = \sup_{t\in K} \left\{ e^{-\lambda t} |x(t) - y(t)|^2 \right\},$$

where λ is a positive real number to be specified later and $\mathcal{K}$ is the set of all compact sub-sets of $\mathbb{R}$.

Note that, for d_K defined above, conditions 1. and 2. of Definition 2.1 are clearly satisfied. Moreover, for all $x, y, z \in \mathbf{E}$ and for every $t \in K \in \mathcal{K}$, by means of Young's inequality, we get:

$$\begin{aligned} |x(t) - y(t)|^2 &\leq \left(|x(t) - z(t)| + |z(t) - y(t)|\right)^2 \\ &\leq 2\left(|x(t) - z(t)|^2 + |z(t) - y(t)|^2\right) \end{aligned}$$

Consequently:

$$e^{-\lambda t}|x(t)-y(t)|^2 \leq 2\left(d_K(x,z)+d_K(z,y)\right)$$

Thus, taking the supremum over K on the left-hand side of the above inequality, we obtain condition 3. of Definition 2.1.

Let $w : \mathcal{K} \longrightarrow \mathcal{K}$ be the mapping defined by:

$$w(K)=\begin{cases} K, & \text{if } K\subset \mathbb{R}_-=]-\infty,0], \\ \\ [0,\ K^*], & \text{otherwise,} \end{cases} \tag{4.2}$$

where $K^*=\sup K$.

Let us now consider the following assumptions:

($\mathcal{B}1$) f is a positive function, non-decreasing with respect to the second and third arguments, and for some real valued function W defined on $\mathbb{R}_+$, the following inequality holds:

$$|f(t,x(t),gx(t))-f(t,y(t),gy(t))| \leq \sqrt{d_{w(K)}(x,y)\, e^{\lambda t}\, W(t)}, \tag{4.3}$$

for all $x,y\in\mathbf{E}$, $t\in K_+$ and $\lambda\geq 0$.

($\mathcal{B}2$) There exist $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $\mu>1$ such that for all $\lambda\geq 0$, the following hold:

(i) $\mathcal{R}_\mu(\lambda) := \displaystyle\int_0^{+\infty} e^{\frac{-p\lambda\tau}{\mu}}\, W^{\frac{p}{2}}(\tau)\, d\tau < \infty$;

(ii) $\forall t>0,\ \mathcal{S}_{\mu,t}(\lambda) := \displaystyle\int_0^t G^q(t,\tau) e^{-\frac{\lambda q}{2}\left[t-\left(\frac{\mu+2}{\mu}\right)\tau\right]}\, d\tau < \infty$;

(iii) $\mathcal{R}_\mu(\lambda)\,\mathcal{S}_{\mu,t}(\lambda) \xrightarrow[\lambda\to\infty]{} 0, \quad \forall t>0.$

($\mathcal{B}3$) For every $x,y\in\mathbf{E}$ such that $x(t)=y(t)$ for $t\leq 0$, if $x(t)\leq y(t)$ for $t>0$, then $gx(t)\leq gy(t)$.

Theorem 3.

Under assumptions ($\mathcal{B}1$)-($\mathcal{B}3$), *the problem* (4.1) *has at last one global solution in* $\mathbf{E}$.

Proof.

Let $F:\mathbf{E}\longrightarrow\mathbf{E}$ be the mapping defined by:

$$Fx(t)=\begin{cases} \varphi(0)+\displaystyle\int_0^t G(t,\tau)f(\tau,x(\tau),gx(\tau))d\tau, & t>0 \\ \\ \varphi(t), & t\leq 0. \end{cases} \tag{4.4}$$

The solutions of (4.1) are the fixed points of F.

Let $\alpha : \mathbf{E} \times \mathbf{E} \longrightarrow \mathbb{R}_+$ be the function defined by:

$$\alpha(x,y) = \begin{cases} 1 : \ x(t) \leq y(t) : \forall t > 0 \ \text{ and } x(t) = y(t) = \varphi(t) : \ t \leq 0 \\ \\ 0 : \ \text{otherwise.} \end{cases}$$

Let us check the generalized $(\boldsymbol{\alpha}_\nu,\boldsymbol{\Psi}^s,w)$ contraction condition (3.2), where $\{\alpha_K\}_{K\in\mathcal{K}} = \{\alpha\}$ and $\{\psi_K\}_{K\in\mathcal{K}}$ is the family of functions ψ_K defined by (4.8).

The following obvious fact is necessary for the final conclusion.

$$\forall x, y \in \mathbf{E} \text{ s.t. } \alpha(x,y) = 0, \quad \alpha(x,y)\, d_K(Fx, Fy) = 0, \ \forall K \in \mathcal{K} \tag{4.5}$$

Let now $x, y \in \mathbf{E}$ such that $\alpha(x,y) = 1$. For $K \in \mathcal{K}$ and $t \in K$ such that $t \leq 0$. We have:

$$|Fx(t) - Fy(t)| = |\varphi(t) - \varphi(t)| = 0.$$

Hence, for all $t \in K$ such that $t \leq 0$ we have

$$e^{-\lambda t}\ |Fx(t) - Fy(t)|^2 = 0. \tag{4.6}$$

Now, for $t \in K$ such that $t > 0$, using $(\mathcal{B}1)$ we obtain:

$$\begin{aligned} |Fx(t) - Fy(t)| &\leq \int_0^t G(t,\tau)\, |f(\tau, x(\tau), gx(\tau)) - f(\tau, y(\tau), gy(\tau))|\, d\tau \\ &\leq \sqrt{d_{w(K)}(x,y)} \int_0^t G(t,\tau) \sqrt{e^{\lambda\tau} W(\tau)}\, d\tau. \end{aligned}$$

Now, multiplying the above inequality by $e^{-\frac{\lambda t}{2}}$, we get:

$$\begin{aligned} e^{-\frac{\lambda t}{2}}\, |Fx(t) - Fy(t)| &\leq \sqrt{d_{w(K)}(x,y)} \left[\int_0^t e^{-\frac{\lambda t}{2}} G(t,\tau)\, e^{\frac{\lambda\tau}{2}} \sqrt{W(\tau)}\, d\tau\right] \\ &= \sqrt{d_{w(K)}(x,y)} \left[\int_0^t e^{-\frac{\lambda t}{2}} G(t,\tau)\, e^{\frac{\lambda(\mu+2)\tau}{2\mu}}\, e^{\frac{-\lambda\tau}{\mu}} \sqrt{W(\tau)}\, d\tau,\right]^2, \end{aligned}$$ where

μ is the constant introduced in $(\mathcal{B}2)$.

In view of $(\mathcal{B}2(i).(ii))$, Hölder's inequality gives:

$$\begin{aligned} e^{-\frac{\lambda t}{2}}\, |Fx(t) - Fy(t)| &\leq \sqrt{d_{w(K)}(x,y)} \left(\int_0^t e^{\frac{-p\lambda\tau}{\mu}}\, W^{\frac{p}{2}}(\tau)\, d\tau\right)^{\frac{1}{p}} \times \left(\int_0^t G^q(t,\tau)\, e^{-\frac{\lambda q}{2}\left[t - \left(\frac{\mu+2}{\mu}\right)\tau\right]}\, d\tau\right)^{\frac{1}{q}} \\ &= \sqrt{d_{w(K)}(x,y)}\mathcal{R}_\mu^{\frac{1}{p}}(\lambda)\mathcal{S}_{\mu,t}^{\frac{1}{q}}(\lambda). \end{aligned}$$

In conclusion, for all $t \in K$ such that $t > 0$ we have:

$$e^{-\lambda t}\, |Fx(t) - Fy(t)|^2 \leq d_{w(K)}(x,y)\ \mathcal{R}_\mu^{\frac{2}{p}}(\lambda)\ \mathcal{S}_{\mu,t}^{\frac{2}{q}}(\lambda). \tag{4.7}$$

Let us now define the function $\psi_K : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ as follows:

$$\psi_K(t) = \begin{cases} \mathcal{R}_\mu^{\frac{2}{p}}(\lambda)\, \mathcal{S}_{\mu,K^*}^{\frac{2}{q}}(\lambda)\, t, & K^* > 0 \\ \\ 0, & K^* \leq 0, \end{cases} \tag{4.8}$$

where, thanks to $(\mathcal{B}2(iii))$ λ is fixed such that

$$2\,\mathcal{R}_\mu^{\frac{2}{p}}(\lambda)\, \mathcal{S}_{\mu,K^*}^{\frac{2}{q}}(\lambda) < 1. \tag{4.9}$$

It is clear that ψ_K satisfies $(\mathbf{\Psi^s}_1)$, $(\mathbf{\Psi^s}_2)$ and $(\mathbf{\Psi^s}_4)$. Furthermore, $\dfrac{\psi_K(s\,.)}{.}$ is constant, thus non-decreasing and in view of (4.9), it satisfies also $\psi_K(2t) < t$. Consequently, according to Remark 2.5, $\psi_K \in \mathbf{\Psi^s}$.

Combining (4.6) and (4.7) taking into account (4.8), leads to

$$\forall x, y \in \mathbf{E} \text{ s.t. } \alpha(x,y) = 1, \quad \alpha(x,y)\, d_K(Fx, Fy) \leq \psi_K\left(d_{w(K)}(x,y)\right), \ \forall K \in \mathcal{K}. \tag{4.10}$$

Now, (3.2) follows immediately from (4.5) and (4.10). In other means, F is a generalized $(\boldsymbol{\alpha}_\nu,\mathbf{\Psi}^s,w)$ contraction.

Condition (I): Let $(x, y) \in \mathbf{E} \times \mathbf{E}$ such that $\alpha(x, y) \geq 1$. Then for $t > 0$, we have $x(t) \leq y(t)$, which implies according to $(\mathcal{B}3)$ that $gx(t) \leq gy(t)$. The following inequality follows so for $t > 0$, from the fact that f is non-decreasing with respect to the second and third arguments

$$\int_0^t G(t,\tau) f(\tau, x(\tau), gx(\tau)) d\tau \leq \int_0^t G(t,\tau) f(\tau, y(\tau), gy(\tau)) d\tau,$$

which clearly leads to $Fx(t) \leq Fy(t)$ for $t > 0$. On the other hand, from (4.4), we have $Fx(t) = Fy(t) = \varphi(t)$ for $t \leq 0$. That is $\alpha(Fx, Fy) \geq 1$, and consequently $(C1)$ is satisfied.

Condition (II): Let $x^0 \in \mathbf{E}$ be the function defined by:

$$x^0(t) = \begin{cases} \varphi(0), & \text{if } \ t > 0 \\ \\ \varphi(t), & \text{if } \ t \leq 0. \end{cases}$$

Since f is positive, then:

$$\int_0^t G(t,\tau) f(\tau, x^0(\tau), gx^0(\tau)) d\tau \geq 0.$$

Hence, for $t > 0$, we have:

$$x^0(t) = \varphi(0) \leq \varphi(0) + \int_0^t G(t,\tau) f(\tau, x^0(\tau), gx^0(\tau)) d\tau = Fx^0(t),$$

and for $t \leq 0$, $\;Fx^0(t) = \varphi(t) = x^0(t)$. That is $\alpha(x^0, Fx^0) \geq 1$.

Furthermore, we have:

$$d_{w^i(K)}(x^0, Fx^0) = d_{w(K)}(x^0, Fx^0) = \sup_{t\in[0,K^*]} e^{-\lambda t}\left|x^0(t) - Fx^0(t)\right|^2 < \infty,$$

for all $i \in \mathbb{N}$. Consequently (P_{C2}) is satisfied.

Note also that:

$$\forall K \in \mathcal{K}, \forall t > 0 : \;\; \psi_K(t) = \psi_{w^i(K)}(t),$$

for all $i \in \mathbb{N}^*$, and so $(C3)$ is satisfied with $\tilde{\psi_K} = \psi_K$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbf{E}$ such that:

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}.$$

That is:

$$x_n(t) \leq x_{n+1}(t), \;\; \text{for all } t > 0 \;\text{ and } x_n(t) = x_{n+1}(t) = \varphi(t) \;\text{ for all }\; t \leq 0. \tag{4.11}$$

Suppose now that $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in \mathbf{E}$, that is:

$$\forall K \in \mathcal{K}, \quad \sup_{t\in K}\left\{e^{-\lambda t}\left|x_n(t) - x(t)\right|^2 \xrightarrow[n\to\infty]{} 0, \right\},$$

which implies that

$$\forall t \in \mathbb{R}, \quad \{x_n(t)\}_{n\in\mathbb{N}} \;\text{ converges to }\; x(t) \;\text{ in }\; \mathbb{R}.$$

Hence, according to (4.11), $\{x_n(t)\}_{n\in\mathbb{N}}$ is a non-decreasing real sequence for $t > 0$ and therefore for all $n \in \mathbb{N}$:

$$x_n(t) \leq x(t), \; \forall t > 0 \;\text{ and }\; x_n(t) = x(t) = \varphi(t), \; \forall t \leq 0.$$

This means that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$ and consequently (P_{C4}) is satisfied.

Then, all conditions of Corollary 3.6 are fulfilled and the proof is complete.

The following Corollary illustrates the efficiency of Theorem 3 in the study of some fractional differential equations with "maxima", namely:

$$\begin{cases} {}^{C}D^{\delta}x(t) = f\left(t, x(t), \max\limits_{\sigma\in[a(t),\, b(t)]} x\left(\sigma\right)\right), & t > 0 \\ x(t) = \varphi(t), & t \leq 0, \end{cases} \tag{4.12}$$

where ${}^{C}D^{\delta}$ denotes the Caputo fractional derivative operator of order $\delta \in \,]0, 1[$, a, b, are real continuous functions defined on $\mathbb{R}_+$ such that $a(t) \leq b(t) \leq t$, $f : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a nonlinear continuous function and $\varphi :]-\infty, 0] \longrightarrow \mathbb{R}$ is a continuous function.

Corollary 4.1.

Assume that the following conditions hold:

($\mathcal{H}_1$) *f is a positive function and non-decreasing with respect to the second and third arguments, such that*

(i) $|f(t,\xi_1,\eta_1) - f(t,\xi_2,\eta_2)| \leq \sqrt{\Upsilon\left(t, |\xi_1-\xi_2|^2, |\eta_1-\eta_2|^2\right)}$, *whenever the left hand side is defined;*

(ii) $\Upsilon : \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ *is a non-decreasing function with respect to the second and third arguments;*

(iii) there exists a real valued function W defined on $\mathbb{R}_+$, such that:

$$\forall z \geq 0: \;\; \Upsilon(., z, z) \leq zW(.).$$

($\mathcal{H}_2$) *There exists $\mu > 1$ such that:*

(i) $\mathcal{R}_\mu(\lambda) := \int_0^{+\infty} e^{-\frac{(1+\delta)\lambda\tau}{\delta\mu}} W^{\frac{1+\delta}{2\delta}}(\tau)\, d\tau < \infty$, *for all* $\lambda > 0$.

(ii) $\mathcal{R}_\mu(\lambda) \xrightarrow[\lambda\to\infty]{} 0, \quad \forall t > 0.$

Then (4.12) *has at least one global solution in* $\mathbf{E}$.

Proof.

Using the properties of fractional integral and derivative operators, problem (4.12) is transformed into the following integral equation, see, e.g. [28, 31, 34, 41–43].

$$x(t) = \begin{cases} \varphi(0) + \int_0^t \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f\left(\tau, x(\tau), \max\limits_{\sigma\in[a(\tau),\, b(\tau)]} x(\sigma)\right) d\tau, & t > 0 \\ \varphi(t), & t \leq 0, \end{cases} \tag{4.13}$$

which is identified to (4.1), with

$$G(t,\tau) = \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} \quad \text{and} \quad gx(t) = \max_{\sigma\in[a(t),\, b(t)]} x(\sigma).$$

Therefore, it is sufficient to show that conditions ($\mathcal{B}1$)-($\mathcal{B}3$) are fulfilled, to deduce then the result from Theorem 3.

Let $x, y \in \mathbf{E}$, $K \in \mathcal{K}$ and $t \in K$ such that $t > 0$. Using ($\mathcal{H}_1(i), (ii)$) we obtain:

$$\left| f(t, x(t), \max_{\sigma\in[a(t),\, b(t)]} x(\sigma)) - f(t, y(t), \max_{\sigma\in[a(t),\, b(t)]} y(\sigma)) \right| \leq$$

$$\sqrt{\Upsilon\left(t, |x(t)-y(t)|^2, \left|\max_{\sigma\in[a(t),\, b(t)]} x(\sigma) - \max_{\sigma\in[a(t),\, b(t)]} y(\sigma)\right|^2\right)} \leq$$

$$\sqrt{\Upsilon\left(t, |x(t)-y(t)|^2, \max_{\sigma\in[a(t),\, b(t)]} |x(\sigma)-y(\sigma)|^2\right)} \leq \sqrt{\Upsilon\left(t, e^{\lambda t} d_{w(K)}(x,y), e^{\lambda t} d_{w(K)}(x,y)\right)},$$

which yields to (4.3) thanks to $(\mathcal{H}_1(iii))$. Consequently, $(\mathcal{B}1)$ is fulfilled. $(\mathcal{H}_2(i))$ implies $(\mathcal{B}2(i))$ with $p = 1+\frac{1}{\delta}$. Let us now check $(\mathcal{B}2(ii))$ where $q = 1+\delta$.

We have:

$$\begin{aligned}\mathcal{S}_{\mu,t}(\lambda) &= \tfrac{1}{\Gamma^q(\delta)} \int_0^t (t-\tau)^{q(\delta-1)} e^{-\frac{\lambda q}{2}\left[t-\left(\frac{\mu+2}{\mu}\right)\tau\right]}\, d\tau \\ &\leq \tfrac{1}{\Gamma^q(\delta)} \int_0^t (t-\tau)^{q(\delta-1)} e^{-\frac{\lambda q}{2}\left(\frac{\mu+2}{\mu}\right)(t-\tau)}\, d\tau.\end{aligned}$$

Performing the change of variable $X = \frac{\lambda q}{2}\left(\frac{\mu+2}{\mu}\right)(t-\tau)$, we get:

$$\begin{aligned}\mathcal{S}_{\mu,t}(\lambda) &\leq \tfrac{1}{\Gamma^q(\delta)} \int_0^\infty \left(\frac{2\mu}{\lambda q(\mu+2)}\right)^{q(\delta-1)} X^{q(\delta-1)} e^{-X}\, dX \\ &= \tfrac{1}{\Gamma^{1+\delta}(\delta)} \left(\tfrac{2\mu}{\lambda q(\mu+2)}\right)^{\delta^2} \Gamma(\delta^2).\end{aligned}$$

Consequently, $(\mathcal{B}2(ii))$ is satisfied and furthermore $\mathcal{S}_{\mu,t}(\lambda) \xrightarrow[\lambda\to\infty]{} 0, \quad \forall t>0.$

The last fact, combined with $(\mathcal{H}_2(ii))$ implies $(\mathcal{B}2(iii))$.

Let $x, y \in \mathbf{E}$, such that $x(t) = y(t)$ for $t \leq 0$. If $x(t) \leq y(t)$ for $t > 0$, then we have:

$$\begin{cases} x(\sigma) \leq y(\sigma), & \sigma \in [a(t), b(t)]_+ \\ x(\sigma) = y(\sigma), & \sigma \in [a(t), b(t)]_-, \end{cases}$$

where $[a(t), b(t)]_+ = [a(t), b(t)] \cap \mathbb{R}_+$ and $[a(t), b(t)]_- = [a(t), b(t)] \cap \mathbb{R}_-$.

Then

$$\begin{cases} \sup\limits_{\sigma\in[a(t),\, b(t)]_+} x(\sigma) \leq \sup\limits_{\sigma\in[a(t),\, b(t)]_+} y(\sigma) \\ \max\limits_{\sigma\in[a(t),\, b(t)]_-} x(\sigma) = \max\limits_{\sigma\in[a(t),\, b(t)]_-} y(\sigma). \end{cases}$$

Consequently $\max\limits_{\sigma\in[a(t),\, b(t)]} x(\sigma) \leq \max\limits_{\sigma\in[a(t),\, b(t)]} y(\sigma)$, that is $(\mathcal{B}3)$ is satisfied.

Then all conditions of Theorem 3 are fulfilled and the proof is complete.

5. Conclusion

In this work, we have introduced a new concept in b-gauge metric spaces called generalized ($\boldsymbol{\alpha}_\nu$,$\boldsymbol{\Psi}^s$,w) contraction, which extended α-ψ contraction in ordinary metric spaces. Some related fixed point results were given using such concept, where weaker conditions have been applied in comparison with existing results. Moreover, applications to delay integral equations on unbounded domain, including fractional differential equations with maxima are provided.

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