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On generalized $(\alpha, *)$-derivations and α-centralizers on Rings

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Abstract. The intention of the current research is to define the concept of $(\alpha, *)$-derivations on ring $\mathcal{R}$, where α is an automorphism of $\mathcal{R}$ and $*$ represents involution on $\mathcal{R}$. We obtain some commutativity theorems in case of prime ring by utilizing the role of α and $*$. We will also discuss the proofs of theorems in case of non-commutative prime ring and under which condition generalized $(\alpha, *)$-derivation behaves like an α-centralizers. Suitable examples are given in favor of introduced concept.

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1. Introduction

Through out the manuscript, the notation $\mathcal{Z}(\mathcal{R})$ stands for the center of an associative ring $\mathcal{R}$. The symbol $[b, d]$ specifies the commutator of $b, d \in \mathcal{R}$, which is represented by the mathematical formula $bd - db$. If $pr = 0$ implies $r = 0$ for every $r \in \mathcal{R}$ and $p > 1$ is a fixed integer, then a ring $\mathcal{R}$ is a p-torsion free ring. A ring $\mathcal{R}$ is a prime if $r\mathcal{R}t = \{0\}$ gives that either $t = 0$ or $r = 0$. It is called semiprime if it fulfills the requirement that $c\mathcal{R}c = \{0\}$ yields that $c = 0$.

In simple terms, the mapping ζ is (skew)-commuting on $\mathcal{R}$ if $\zeta(c)c + c\zeta(c) = 0$ for each of $c \in \mathcal{R}$. If $\zeta(c)c + c\zeta(c) \in \mathcal{Z}(\mathcal{R})$ for each $c \in \mathcal{R}$, then a map ζ from $\mathcal{R}$ to $\mathcal{R}$ is thought to be (skew)-centralizing on $\mathcal{R}$. If the mapping η from $\mathcal{R}$ to $\mathcal{R}$ fulfills the equation $\eta(ce) = \eta(c)e + c\eta(e)$, for each of $c, e \in \mathcal{R}$, then it is regarded as a derivation on $\mathcal{R}$.

Let $\mathcal{R}$ be a ring whose automorphism is β. A map h on $\mathcal{R}$ satisfying $\mathrm{h}(dk) = \mathrm{h}(d)\beta(k) + d\mathrm{h}(k)$ is recognized as the β-derivation (skew-derivation) if it holds for any

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pair d, k in $\mathcal{R}$ and h has additivity. The combination form $\mathrm{h} = \beta - \mathcal{I}$ served as the β-derivation if we symbolize the identity map on $\mathcal{R}$ by $\mathcal{I}$.

Before moving onto this section's primary findings, we clarify a few fundamental concepts and terminologies. Involution is an additive mapping defined as $*$ from $\mathcal{R}$ to $\mathcal{R}$ that fulfills the following two requirements: $(dj)^* = j^*d^*$ and $(d^*)^* = d$ for each $d, j \in \mathcal{R}$. Invertible matrices and identity matrices are the most common examples of involution over the matrix ring. A $*$-ring, sometimes referred to as an involution ring (or ring combined with an involution $*$). The references [1], [2], [3], [4], [5], [6] are ideal places for start reading about generalized derivations, involution, centralizers and their related topics.

A ring possessing involution $*$ is called a $*$-prime ring if $aRb = aRb^* = \{0\}$, or $aRb = a^*Rb = \{0\}$, where $a, b \in R$, implies that either $a = 0$ or $b = 0$. It is a noticeable fact that all prime rings possessing involution $*$ are $*$-prime but not necessarily prime. For example, R^0 denotes the opposite ring of a prime ring R, then $R \times R^0$ having exchange involution $*_{xe}$ defined as

$$*_{xe}(x, y) = (y, x)$$

is a $*_{xe}$-prime but not prime.

Let R be a $*$-ring. A mapping $D : R \to R$ is said to be a $*$-derivation if it satisfies: (i) Additivity and (ii) $D(xy) = D(x)y^* + xD(y)$ for all $x, y \in R$. In the case where R is a commutative $*$-ring, D has the form $D(x) = a(x - x^*)$ for some $a \in R$, which is a $*$-derivation on R.

Following [7], a mapping $T : R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$ and T is also additive. In the same line of investigation, the expression for $*$-centralizer comes out as follows: A mapping T on R, additive and satisfying $T(xy) = T(x)y^*$ and $T(xy) = x^*T(y)$ for all $x, y \in R$ will be called left $*$-centralizer and right $*$-centralizer respectively on R. A remarkable investigation on the theory of centralizers and $*$-centralizers presented in [8–11]. In [4], authors proved an advancement of the generalized concept of $*$-derivation on standard operator algebra.

A ring with endomorphism α, if we take $\gamma = \varsigma - \alpha$, then γ is an (α, I)-derivation, but not a derivation when R is semiprime and $I = \alpha$. The inclusive information can be found in [12]. Some commutativity results about $*$-bimultipliers and generalized $*$-biderivations can be viewed in [3]. We review the concept of such γ and introduce the concept of $(\alpha, *)$-derivation and generalized $(\alpha, *)$-derivation on R as follows:

Definition 1. *Let* $\mathrm{D} : \mathcal{R} \longrightarrow \mathcal{R}$ *be an additive map.* D *is said to be* $(\alpha, *)$*-derivation on* $\mathcal{R}$ *if it satisfy the conditions:*

$$\mathrm{D}(\nu k) = \mathrm{D}(\nu)\alpha(k) + \nu^*\mathrm{D}(k), \textit{ for every } \nu, k \in \mathcal{R},$$

where $*$ is an involution on $\mathcal{R}$ and α is the automorphism on $\mathcal{R}$.

Example 1. *Consider a $*$-ring $R = \left\{ \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \mid p, q, r \in 2\mathbb{Z}_4 \right\}$. Define involution mapping $*$ from R to itself by $\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}^* = \begin{pmatrix} -p & 0 \\ 0 & 0 \end{pmatrix}$ for all $p \in 2\mathbb{Z}_4$ under matrix addition and matrix multiplication, where $\mathbb{Z}_4$ has its usual notation. Take a mapping $\alpha, \mathcal{D} : R \to R$ defined by $\alpha\left[\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\right] = \begin{pmatrix} r & 0 \\ q & p \end{pmatrix}$ and $\mathcal{D}\left[\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}$ for all $p, q, r \in 2\mathbb{Z}_4$. It is clear that $\mathcal{D}$ satisfy the above definition, hence it is $(\alpha, *)$-derivation on R.*

We observe that if $* = \mathcal{I}$, the definition 1 will be set as skew derivation with automorphism α. It is somewhat a unified notion of skew derivation and $*$-derivation. Next we extend our definition to the case of generalized derivation.

Definition 2. *Let $\mathrm{F}, \mathrm{D} : \mathcal{R} \longrightarrow \mathcal{R}$ be additive maps. F is said to be generalized $(\alpha, *)$-derivation associated with D on $\mathcal{R}$ if it satisfy the below condition*

$$\mathrm{F}(\nu k) = \mathrm{F}(\nu)\alpha(k) + \nu^* \mathrm{D}(k), \text{ for every } \nu, k \in \mathcal{R},$$

where $*$ is an involution on $\mathcal{R}$ and α is the automorphism on $\mathcal{R}$.

Example 2. *Let*

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in 2\mathbb{Z}_4 \right\}$$

be a ring with usual operation of matrix addition and multiplication.

Define $\mathrm{F}, \mathrm{D} : R \to R$ as

$$\mathrm{F}(r) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and a map D is given by

$$D(r) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, r \in R.$$

Define $\alpha : R \to R$ by

$$\alpha(r) = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}.$$

The involution is given by

$$r^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

F *will be a generalized $(\alpha, *)$-derivation associated with* D.

A lot of research has been done in the context of involution involved with derivation, generalized derivation, Jordan derivation, left derivation, etc. Our present research is motivated by all the above theories and the role of automorphism on R and involution. We will prove some commutativity theorems in the setting of prime and semiprime rings. We will observe that α and $*$ play a crucial role in our proofs. The commutativity theorem on prime rings possessing automorphisms (or endomorphisms) proved in [5, 6, 9, 13]. Further we refer the reader to the extensive bibliography contained in it.

Motivated by the above literature review and concepts, we putout the extension of the notion of generalized $(\alpha, *)$-derivation to generalized $(\alpha, *)$-n-derivation on rings as follows:

Definition 3. *A mapping $F : R^n \to R$ is called a generalized $(\alpha, *)$-n -derivation if there exists an $(\alpha, *)$-n-derivation $D : R^n \to R$ such that*

$$F(\varsigma_1, \ldots, \varsigma_k\varsigma_k', \ldots, \varsigma_n) = F(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(\varsigma_k') + \varsigma_k^* D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n)$$

for all $\varsigma_1, \ldots, \varsigma_k, \varsigma_k', \ldots, \varsigma_n \in R$.

2. Results on prime $*$-ring

We start with the following results:

Lemma 1. *[14] The center of $\mathcal{R}$ includes the center of a nonzero ideal (one-sided) for a semiprime ring $\mathcal{R}$. Any commutative ideal (one-sided) is immediately enclosed $Z(\mathcal{R})$.*

Theorem 1. *Let a semiprime $*$-ring be $\mathcal{R}$, $*$ be an involution, and α be an automorphism on $\mathcal{R}$. If $\mathrm{F}_1, \mathrm{F}_2$ are two generalized $(\alpha, *)$-n-derivations on $\mathcal{R}$ associated with $(\alpha, *)$-n-derivation $\mathrm{D}_1, \mathrm{D}_2$ respectively, such that*

$$\mathrm{F}_1(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \mathrm{F}_2(\varsigma_1, \varsigma_2, ..., \varsigma_n)$$

for all $\varsigma_1, \varsigma_2, ..., \varsigma_n \in \mathcal{R}$, then $\mathrm{D}_1 = \mathrm{D}_2$.

Proof. By the hypothesis, we are given that

$$\mathrm{F}_1(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \mathrm{F}_2(\varsigma_1, \varsigma_2, ..., \varsigma_n), \text{ for every } \varsigma_1, \varsigma_2, ..., \varsigma_n \in \mathcal{R}. \tag{1}$$

Rewrite (1) to get the form

$$\mathrm{F}_1(\varsigma_1, ..., \varsigma_k\varsigma_k', ..., \varsigma_n) = \mathrm{F}_2(\varsigma_1, ..., \varsigma_k\varsigma_k', ...,_n), \text{ for every } \varsigma_1, \varsigma_2, ..., \varsigma_k, \varsigma_k', ..., \varsigma_n \in \mathcal{R}. \tag{2}$$

By definition the above equation reword as

$$\begin{aligned}\mathrm{F}_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(\varsigma_k') + \varsigma_k^* D_1(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) &= \mathrm{F}_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(\varsigma_k') \\ &\quad + \varsigma_k^* D_2(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n).\end{aligned}$$

Application of (1) with the last expression to find

$$\begin{aligned} \varsigma_k^* D_1(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n) &= \varsigma_k^* D_2(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n) \\ &= \varsigma_k^* \left(D_1(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n) - D_2(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)\right) \\ &= 0, \text{ for each } \varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n \in \mathcal{R}. \end{aligned} \tag{3}$$

Particularly consider $\varsigma_k = \varsigma_k^*$ for the case of $* - ring$ to obtain

$$\varsigma_k R\left(D_1(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n) - D_2(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)\right) = 0 \text{ for each } \varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n \in \mathcal{R}.$$

Condition of semiprimeness of R implies that

$$D_1(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n) = D_2(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n), \text{ for every } \varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n \in \mathcal{R}$$

Therefore, $D_1 = D_2$. This completes the proof. □

Theorem 2. *If a prime $*$-ring R admits a nonzero $(\alpha,*)$-n-derivation D, then R is commutative.*

Proof. Given that D is a $(\alpha,*)$-n-derivation, by definition we have

$$D(\varsigma_1,\ldots,\varsigma_k\varsigma_k',\ldots,\varsigma_n) = D(\varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n)\alpha(\varsigma_k') + \varsigma_k^* D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)$$

for every $\varsigma_k,\varsigma_k' \in R$. Now substitute $\varsigma_k = \varsigma_k y$, where $y \in R$, we get

$$D(\varsigma_1,\ldots,\varsigma_k y\varsigma_k',\ldots,\varsigma_n) = D(\varsigma_1,\ldots,\varsigma_k y,\ldots,\varsigma_n)\alpha(\varsigma_k') + (\varsigma_k y)^* D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)$$

for every $\varsigma_k,\varsigma_k',y \in R$. Expanding the terms,

$$\begin{aligned} D(\varsigma_1,\ldots,\varsigma_k y\varsigma_k',\ldots,\varsigma_n) = {} & D(\varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n)\alpha(y)\alpha(\varsigma_k') \\ & + \varsigma_k^* D(\varsigma_1,\ldots,y,\ldots,\varsigma_n)\alpha(\varsigma_k') \\ & + y^*\varsigma_k^* D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n) \end{aligned} \tag{4}$$

for every $\varsigma_1,..,\varsigma_k,\varsigma_k',...,\varsigma_n,y \in R$. Alternative expression for the left hand side of above equation is given by

$$\begin{aligned} D(\varsigma_1,\ldots,\varsigma_k(y\varsigma_k'),\ldots,\varsigma_n) = {} & D(\varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n)\alpha(y\varsigma_k') \\ & + \varsigma_k^* D(\varsigma_1,\ldots,y\varsigma_k',\ldots,\varsigma_n), \end{aligned}$$

for every $\varsigma_1,\ldots,\varsigma_k,\varsigma_k',\ldots,\varsigma_n,y,\varsigma_k^* \in R$. Expanding the terms

$$\begin{aligned} D(\varsigma_1,\ldots,\varsigma_k(y\varsigma_k'),\ldots,\varsigma_n) = {} & D(\varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n)\alpha(y)\alpha(\varsigma_k') \\ & + \varsigma_k^* D(\varsigma_1,\ldots,y,\ldots,\varsigma_n)\alpha(\varsigma_k') \\ & + \varsigma_k^* y^* D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n). \end{aligned} \tag{5}$$

Substituting equation (4) into equation (5), we get

$$0 = (\varsigma_k^* y^* - y^*\varsigma_k^*) D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n). \quad \text{for every } \varsigma_1,...,\varsigma_n,y \in R.$$

This simplifies to

$$0 = [\varsigma_k^*, y^*]D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n).$$

Now, substitute $\varsigma_k^* = \varsigma_k$ and $y^* = y$ into the above equation,

$$[\varsigma_k, y]D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) = 0 \quad \text{for every} \varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n \in R.$$

Replace y by yr to obtain

$$[\varsigma_k, y]rD(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) = 0 \text{ for every } \varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n, y, r \in R. \tag{6}$$

By primeness, we conclude that either

$$D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) = 0 \quad \text{for every } \varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n \in R$$

or

$$[\varsigma_k, y] = 0 \text{ for every } \varsigma_k, y \in R.$$

Since $D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) \neq 0$, it follows that

$$[\varsigma_k, y] = 0 \text{ for every } \varsigma_k, y \in R.$$

Thus, R is commutative. □

Corollary 1. *If a non-commutative prime $*$-ring R admits a $(\alpha, *)$-n-derivation D, then D =0.*

Theorem 3. *Let R be a prime $*$-ring. If R admits a nonzero generalized $(\alpha, *)$-n-derivation F associated with an $(\alpha, *)$-n-derivation D, then one of the conditions hold:*
1. R is commutative.
2. F acts as left α-centralizer.

Proof. Since F is a generalized $(\alpha, *)$-n-derivation, then we obtain for all $\varsigma_1, ..., \varsigma_k, \varsigma_k', ..., \varsigma_n \in R$

$$\mathrm{F}(\varsigma_1, ..., \varsigma_k y \varsigma_k', ..., \varsigma_n) = \mathrm{F}(\varsigma_1, \ldots, \varsigma_k y, \ldots, \varsigma_n)\alpha(\varsigma_k') + (\varsigma_k y)^* D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n). \tag{7}$$

Simplify above expression to get

$$\begin{aligned} \mathrm{F}(\varsigma_1, ..., \varsigma_k y \varsigma_k', ..., \varsigma_n) \quad = \quad & \mathrm{F}(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(y)\alpha(\varsigma_k') \\ & + (\varsigma_k)^* D(\varsigma_1, \ldots, y, \ldots, \varsigma_n)\alpha(\varsigma_k') \\ & + y^* \varsigma_k^* D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) \end{aligned} \tag{8}$$

for every $\varsigma_1, ..., \varsigma_k, y, \varsigma_k', ..., \varsigma_n$ in R. Alternatively in view of (7) we find $F(\varsigma_1, \ldots, \varsigma_k y \varsigma_k', \ldots, \varsigma_n) = F(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(y\varsigma_k') + \varsigma_k^* D(\varsigma_1, \ldots, y\varsigma_k', \ldots, \varsigma_n)$. Expand the right hand side of last expression to get

$$F(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(y)\alpha(\varsigma_k') + \varsigma_k^* D(\varsigma_1, \ldots, y, \ldots, \varsigma_n)\alpha(\varsigma_k') + \varsigma_k^* y^* D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) \tag{9}$$

Combining (15) and (9) together, we get for every $\varsigma_1, ..., \varsigma_k, y, \varsigma_k', ..., \varsigma_n$ in R

$$y^*\varsigma_k^* D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) = \varsigma_k^* y^* D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) \tag{10}$$

Put ς_k and y instead of ς_k^* and y^* to observe that

$$[\varsigma_k, y](D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n)) = 0 \text{ for every } \varsigma_1, ..., \varsigma_k, y, \varsigma_k', ..., \varsigma_n \in R. \tag{11}$$

Again replace y by yr where $r \in R$ in (11) and using it

$$[\varsigma_k, y]R(D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n)) = 0 \text{ for every } \varsigma_1, ..., \varsigma_k, y, \varsigma_k', ..., \varsigma_n \in R. \tag{12}$$

From (12), we say that R can be written as $K_1^+ \cup K_2^+$, where

$$K_1^+ = \{[\varsigma_k, y] = 0 \mid \varsigma_k, y \in R\}$$

and

$$K_2^+ = \{\varsigma_1 ..., \varsigma_n \in R \mid D(\varsigma_1, \ldots, \varsigma_k', \ldots, \varsigma_n) = 0\}.$$

Which is a contradiction to the fact that R cannot be determined by the union of two additive subgroups, namely K_1^+ and K_2^+. Hence, primeness implies that either $K_1^+ = R$ or $K_2^+ = R$.

If $K_1^+ = R$, then R is commutative by Lemma 1. In case $K_2^+ = R$, we say that after simple manipulation

$$[D(\varsigma_1, \ldots, \varsigma_n), r] = 0 \quad \text{for every } r \in R.$$

Hence, D commutes with R. An application of Theorem 2 guarantees that either $D = 0$ or R is commutative. Again, we are done in the second case. On the other hand take $D = 0$, and use the definition to get the expression

$$F(\varsigma_1, \ldots, \varsigma_k r, \ldots, \varsigma_n) = F(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(r)$$

for every $\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n, r \in R$, where F acts as a left α-centralizer.

Theorem 4. *Let R be a non-commutative prime $*$-ring. If R admits a nonzero generalized $(\alpha, *)$-n-derivation F associated with an $(\alpha, *)$-n-derivation D, then F acts as left α-centralizer.*

Proof. The proof is straight forward by the application of Theorem 3.

Theorem 5. *Let R be a 2-torsion-free prime $*$-ring having generalized $(\alpha, *)$-n-derivations F_1 and F_2 associated with $(\alpha, *)$-n-derivations D_1 and D_2 respectively. If*

$$\begin{gathered} F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)D_2(y_1, \ldots, y_k, \ldots, y_n) \\ -F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)D_1(y_1, \ldots, y_k, \ldots, y_n) = 0, \end{gathered}$$

for each $\varsigma_1, ..., \varsigma_n, y_1, ..., y_n \in R$, then one of the following holds:

(i) $F_1 = 0$ *or* F_2 *acts as a left* α*-centralizer.*

(ii) $F_2 = 0$ *or* F_1 *acts as a left* α*-centralizer.*

Proof. Suppose that

$$\begin{aligned} &F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)D_2(y_1, \ldots, y_k, \ldots, y_n) \\ &-F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)D_1(y_1, \ldots, y_k, \ldots, y_n) = 0. \end{aligned} \tag{13}$$

Put $y_k z$ in place of y_k, we have for each $\varsigma_1, ..., \varsigma_n, y_1, ..., y_n \in R$

$$\begin{aligned} &F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)D_2(y_1, \ldots, y_k z, \ldots, y_n) \\ &-F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)D_1(y_1, \ldots, y_k z, \ldots, y_n) = 0. \end{aligned}$$

Explore the above equation

$$\begin{aligned} &F_1\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\{D_2(y_1, \ldots, y_k, \ldots, y_n)\alpha(z) + y_k^* D_2(y_1, \ldots, z, \ldots, y_n)\} \\ &- F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\{D_1(y_1, \ldots, y_k, \ldots, y_n)\alpha(z) + y_k^* D_1(y_1, \ldots, z, \ldots, y_n)\} = 0. \end{aligned}$$

From (13), we arrive at

$$\begin{aligned} &F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k^* D_2(y_1, \ldots, z, \ldots, y_n) \\ &-F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k^* D_1(y_1, \ldots, z, \ldots, y_n) = 0, \end{aligned} \tag{14}$$

for each $\varsigma_1, ..., \varsigma_n, y_1, ..., y_n \in R$. Multiplying (14) from the right by $pD_1(y'_1, \ldots, y'_k, \ldots, y'_n)$ where $p, y'_k \in R$, we obtain

$$\begin{aligned} &(F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k^* D_2(y_1, \ldots, z, \ldots, y_n) \\ &-F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k^* D_1(y_1, \ldots, z, \ldots, y_n))pD_1(y'_1, \ldots, y'_k, \ldots, y'_n) = 0, \end{aligned} \tag{15}$$

for each $\varsigma_1, ..., \varsigma_n, y_1, ..., y_n$ in R.

Case 1 In view of (14), (15) takes the form

$$2F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k D_1(y_1, \ldots, z, \ldots, y_n)pD_1(y'_1, \ldots, y'_k, \ldots, y'_n) = 0$$

Using $*$-primeness of R and 2-torsion-freeness of R, we find

$$F_2(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k D_1(y_1, \ldots, z, \ldots, y_n) = 0, \quad \text{for every } \varsigma_1, ..., \varsigma_n, y_1, ..., y_n, z \in R.$$

Again, making use of primeness, we can conclude either $F_2 = 0$ or $D_1 = 0$. In case $D_1 = 0$, we obtain

$$F_1(\varsigma_1, \ldots, \varsigma_k\varsigma'_k, \ldots, \varsigma_n) = F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)\alpha(\varsigma'_k),$$

which implies that F_1 acts as an α-centralizer on R.

Case 2 Multiply (15) by $pD_2(y'_1, \ldots, y'_k, \ldots, y'_n)$ from the right and use it again to obtain

$$2F_1(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)y_k D_2(y_1, \ldots, z, \ldots, y_n)pD_2(y'_1, \ldots, y'_k, \ldots, y'_n) = 0,$$

for each $\varsigma_1, ..., \varsigma_n, y_1, ..., y_n$ in R. In view of (15), the equation takes the form After repeating the similar footsteps as in case 1, we conclude either $F_1 = 0$ or $D_2 = 0$. In case $D_2 = 0$, F_2 acts as a left α-centralizer. □

Theorem 6. *Let R be a semi-prime $*$-ring admitting a generalized $(\alpha,*)$-n-derivation F with associated $(\alpha,*)$-n-derivation D. Then*

$$D(R,R,\ldots,R)\subseteq Z(R).$$

Proof. Since R is a $*$-ring and admits a generalized $(\alpha,*)$-n-derivation F, we take a quick start from equation (6) in Theorem 2

$$[x,y]RD(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)=0 \text{ for every } \varsigma_1,...,\varsigma_n\in R. \tag{16}$$

Replacing x by $D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)x$, we obtain for every $\varsigma_1,...,\varsigma_n\in\mathcal{R}$

$$\begin{aligned}D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)[x,y]RD(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)\\+[D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n),y]xRD(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)=0.\end{aligned}\tag{17}$$

From (16) and (17), we conclude

$$[D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n),y]xRD(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)=0 \text{ for every } \varsigma_1,...,\varsigma_n\in\mathcal{R}. \tag{18}$$

we can write also

$$[D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n),y]xRyD(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)=0 \text{ for every } \varsigma_1,...,\varsigma_n,y,x\in\mathcal{R}. \tag{19}$$

Now multiply (18)by y from right and subtract the resulting equation with (19)to find

$$[D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n),y]xR[D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n),y]=0 \text{ for every } \varsigma_1,...,\varsigma_n,x,y\in\mathcal{R}.$$

Since this holds for all $y\in R$, it follows that

$$[D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n),y]=0 \quad \text{for every } y\in R.$$

Thus, we conclude that

$$D(\varsigma_1,\ldots,\varsigma_k',\ldots,\varsigma_n)\subseteq Z(R).$$

□

Theorem 7. *Let R be a semiprime ring with involution $*$. If F is a generalized $(\alpha,*)$-n-derivation of R associated with an $(\alpha,*)$-n-derivation D such that*

$$F(\varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n)y_i=\varsigma_iF(y_1,\ldots,y_k,\ldots,y_n)$$

for every $\varsigma_1,\ldots,\varsigma_k,\ldots,\varsigma_n,y_1,\ldots,y_k,\ldots,y_n\in R$, then F is a left α-centralizer.

Proof. By the given hypotheses, we have

$$F(\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n)y_i = \varsigma_i F(y_1, \dots, y_k, \dots, y_n) \tag{20}$$

for every $\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n, y_1, \dots, y_k, \dots, y_n \in R$. Substituting $y_k = y_k z$, where $z \in R$, we obtain

$$F(\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n)y_i = \varsigma_i F(y_1, \dots, y_k z, \dots, y_n).$$

Using definition on the right hand side, we have

$$F(\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n)y_i = \varsigma_i F(y_1, \dots, y_k, \dots, y_n)\alpha(z) + \varsigma_i y_k^* D(y_1, \dots, z, \dots, y_n).$$

Since α is an automorphism, we may put $z = \alpha^{-1}(z)$ to have

$$F(\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n)y_i = \varsigma_i F(y_1, \dots, y_k, \dots, y_n)z + \varsigma_i y_k^* D(y_1, \dots, \alpha^{-1}(z), \dots, y_n).$$

Rearranging by putting $y_i = y_i z$ in the above equation, we get

$$F(\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n)y_i - \varsigma_i F(y_1, \dots, y_k, \dots, y_n)z = \varsigma_i y_k^* D(y_1, \dots, \alpha^{-1}(z), \dots, y_n),$$

for every $\varsigma_1, \dots, \varsigma_k, \dots, \varsigma_n, y_1, \dots, y_k, \dots, y_n \in R$. Taking z as a common factor and using (20), we have

$$\varsigma_i y_k^* D(y_1, \dots, \alpha^{-1}(z), \dots, y_n) = 0, \text{for every } \varsigma_i, y_1, \dots, y_k, \dots, y_n \in R.$$

It follows that

$$\varsigma_i y_k D(y_1, \dots, z, \dots, y_n) = 0. \quad \text{for every } \varsigma_i, y_1, \dots, y_k, \dots, y_n \in R.$$

A simple manipulation yields that

$$D(y_1, \dots, z, \dots, y_n)\varsigma_i D(y_1, \dots, z, \dots, y_n)y_k D(y_1, \dots, z, \dots, y_n)\varsigma_i = 0,$$

for every $\varsigma_i, y_1, .., y_k, z \in R$. Making use of semi-primeness of R, it follows that

$$D(y_1, \dots, z, \dots, y_n) = 0 \quad \text{for every } y_1, .., y_i, z \in R.$$

Hence F acting as a left α-centralizer.

□

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