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Forcing Clique Domination in Graphs

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Abstract. The clique domination number of some special graphs such as paths, cycles, complete graphs, generalized wheels, generalized fans, and complete bipartite graphs is presented. The forcing clique domination number of these graphs, along with binary operations such as join, corona, and lexicographic product of two graphs, is also determined. Connected graphs with forcing clique domination number equal to 0, 1, or a, where a is greater than 1 but less than the clique domination number, are characterized. Necessary and sufficient conditions for the forcing clique domination number to be equal to the clique domination number are given. Since some of the graphs in this study do not have a clique dominating set, the forcing clique domination number is undefined in those cases.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph.For any vertex $t \in V(G)$,the closed neighborhood of t is defined as the set $N_G[t] = \{t\} \cup \{s \in V(G) : st \in E(G)\}$. If T is a nonempty subset of X, then $N_G[T] = \bigcup_{t\in T} N_G[t]$. A nonempty set $T \subseteq V(G)$ is a dominating set of G if for every $u \in V(G)\backslash T$, there exists $t \in T$ such that $tu \in E(G)$, that is, $N_G[T] = V(G)$.

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The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G. A γ-set T of G is a dominating set of G with $|T| = \gamma(G)$.

A graph is complete if every two of its vertices are adjacent. Let G be a nontrivial connected graph. A dominating set C of $V(G)$ is a clique dominating set of G if the induced subgraph $\langle C\rangle$ of C is complete. The minimum cardinality of a clique dominating set of G, denoted by $\gamma_{cl}(G)$,is called the clique domination number of G. A γ_{cl}-set C of G is a clique dominating set of G with $|C| = \gamma_{cl}(G)$. Graph G is considered a $non-\gamma_{cl}-graph$ if it does not contain a clique dominating set, following a similar definition to that of a non-γ_{p0}-graph as in [1].

Let C be a γ_{cl}-set of a graph G. A subset L of C is said to be a *forcing subset* for C if C is the unique γ_{cl}-set containing L. The *forcing clique domination number of* C is given by $f\gamma_{cl}(C) = \min\{|L| : L \text{ is a forcing subset for } C\}$. The *forcing clique domination number of* G is given by

$$f\gamma_{cl}(G) = \min\{f\gamma_{cl}(C) : C \text{ is a } \gamma_{cl}\text{-set of } G\}$$

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex set

$$V(G + H) = V(G) \cup V(H)$$

and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

The *corona* of two graphs G and H, denoted by $G\circ H$, is defined to be the graph obtained by taking one copy of G and $|V(G)|$ copies of H and then forming the joins $\langle v\rangle + H^v = v + H^v$ for each $v \in V(G)$, where H^v is a copy of H correponding to vertex v.

The *lexicographic product* or *composition* of two graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$. Observe that a subset C of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x\in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. We shall use this form to denote any subset C of $V(G[H])$.

The clique domination was investigated in [2] and [3]. The concept of forcing domination was first studied by Chartrand, et al. in [4]. Closed neighborhood, domination number, forcing domination number, the binary operations such as join, corona and lexicographic product of graphs, and other variations of forcing domination can be found in [5],[6],[7],[8] and [9]. Additional basic graph-theoretic terminology can be found in [10].

The forcing clique domination number is important when it comes to fault-tolerant sensor network optimization in smart cities. Sensors are placed in these networks to monitor infrastructure, health, traffic, and air quality. Certain sensor groups naturally form cliques, which are fully connected subgraphs that guarantee effective data sharing. These sensors create graphs with edges that indicate direct communication links. To guarantee smooth network coverage, a clique dominating set ensures that each sensor is either inside a clique or directly connected to one [11]. This structure is improved by the forcing property, which ensures that the activation of a small number of important sensors triggers the activation of others, reducing redundancy and increasing data collection and transmission efficiency [12]. This ensures that the network continues to operate with low resource consumption even in the event that certain sensors fail [13].

In addition to energy efficiency, the forcing clique domination number improves fault tolerance and sensor network resilience. The system can tolerate failures and continue to function by carefully choosing a minimum clique dominating set. This is particularly helpful in fields where dependability is essential, such as emergency response systems, military communication, and disaster monitoring [14].

Example 1.1. *Consider the graph G in Figure 1. It is clear to see that*

$$\begin{aligned} R_1 &= \{x, u_1\}, \\ R_2 &= \{x, u_2\}, \\ R_3 &= \{x, u_3\}, \\ &\vdots \\ R_{m-1} &= \{x, u_{m-1}\}, \text{ and} \\ R_m &= \{x, u_m\} \end{aligned}$$

are γ_{cl}-sets of G. Clearly, for all $i = 1, 2, \ldots, m$, $T_i = \{u_i\}$ is uniquely contained in each γ_{cl}-set R_i of G and so, T_i is a forcing subset for each R_i. Thus, $f\gamma_{cl}(G) = |T_i| = 1$.

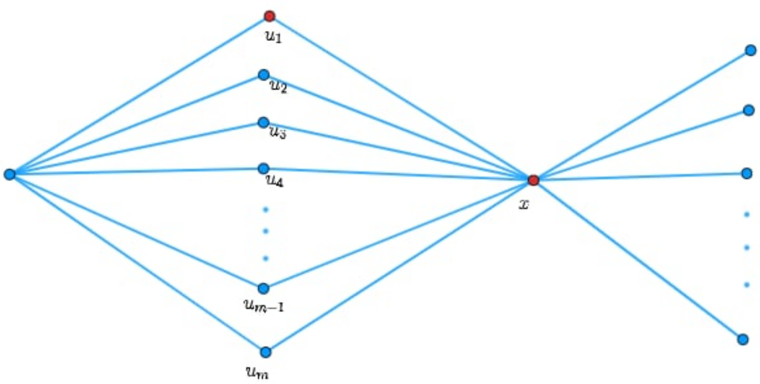

Figure 1: Graph G with $f\gamma_{cl}(G) = 1$.

Example 1.2. *Consider the graph $G[H]$ in Figure 2. Clearly, $\gamma_{cl}(G) = 3$. By Corollary 2.10, $\gamma_{cl}(G[H]) = 3$. It is clear to see that*

$$\begin{array}{lll}
S_1 = \{(a,x),(b,x),(c,x)\}, & S_{10} = \{(a,y),(b,x),(c,x)\}, & S_{19} = \{(a,z),(b,x),(c,x)\},\\
S_2 = \{(a,x),(b,x),(c,y)\}, & S_{11} = \{(a,y),(b,x),(c,y)\}, & S_{20} = \{(a,z),(b,x),(c,y)\},\\
S_3 = \{(a,x),(b,x),(c,z)\}, & S_{12} = \{(a,y),(b,x),(c,z)\}, & S_{21} = \{(a,z),(b,x),(c,z)\},\\
S_4 = \{(a,x),(b,y),(c,x)\}, & S_{13} = \{(a,y),(b,y),(c,x)\}, & S_{22} = \{(a,z),(b,y),(c,x)\},\\
S_5 = \{(a,x),(b,y),(c,y)\}, & S_{14} = \{(a,y),(b,y),(c,y)\}, & S_{23} = \{(a,z),(b,y),(c,y)\},\\
S_6 = \{(a,x),(b,y),(c,z)\}, & S_{15} = \{(a,y),(b,y),(c,z)\}, & S_{24} = \{(a,z),(b,y),(c,z)\},\\
S_7 = \{(a,x),(b,z),(c,x)\}, & S_{16} = \{(a,y),(b,z),(c,x)\}, & S_{25} = \{(a,z),(b,z),(c,x)\},\\
S_8 = \{(a,x),(b,z),(c,y)\}, & S_{17} = \{(a,y),(b,z),(c,y)\}, & S_{26} = \{(a,z),(b,z),(c,y)\}, \text{ and}\\
S_9 = \{(a,x),(b,z),(c,z)\}, & S_{18} = \{(a,y),(b,z),(c,z)\}, & S_{27} = \{(a,z),(b,z),(c,z)\}
\end{array}$$

are γ_{cl}-sets of $G[H]$. Clearly, there exists no subset with 1 and 2 vertices that it is contained in a unique γ_{cl}-set of $G[H]$. Thus, for all $i = 1, 2, \ldots, 27$, S_i is a forcing subset for itself and so, $f\gamma_{cl}(G[H]) = |S_i| = 3 = \gamma_{cl}(G[H])$.

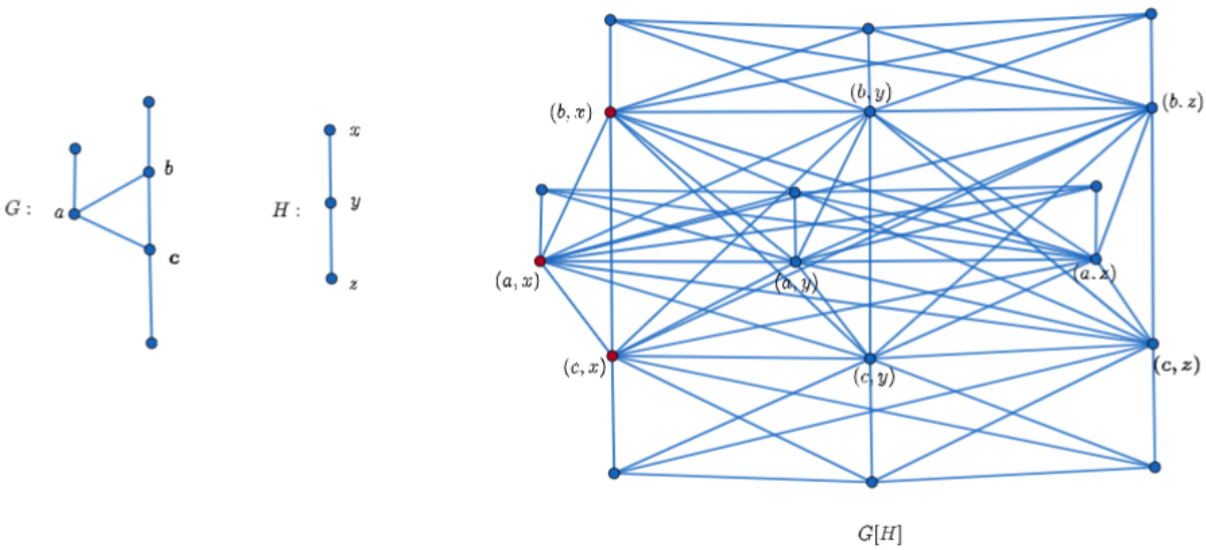

Figure 2: Graph $G[H]$ with $f\gamma_{cl}(G[H]) = 3$.

2. Known Results

This section presents known results on the domination number and the clique domination number of a graph G, and of graphs resulting from some binary operations.

Proposition 2.1. [15] *For $n \geq 3$,*

$$\gamma(P_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proposition 2.2. [16] *If n is a positive integer , then $\gamma(K_n) = 1$.*

Theorem 2.3. [2] *Let G be a connected graph. Then $\gamma_{cl}(G) = 1$ if and only if $\gamma(G) = 1$.*

Theorem 2.4. [2] *Let G and H be any two graphs. A subset S of $V(G+H)$ is a clique dominating set of $G+H$ if and only if one of the following statements holds:*

(*i*) *S is clique dominating set of G*

(*ii*) *S is a clique dominating set of H.*

(*iii*) *$S = S_1 \cup S_2$, where $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are cliques in G and H, respectively.*

Corollary 2.5. [2] *Let G and H be nontrivial graphs . Then*

$$\gamma_{cl}(G+H) = \begin{cases} 1, & if\ \gamma(G) = 1\, or\ \gamma(H) = 1 \\ 2, & otherwise \end{cases}$$

Theorem 2.6. [3] *If G is a finite graph that is connected and has no induced P_5 or C_5, then G has a clique dominating set.*

Theorem 2.7. [2] *Let G be a connected nontrivial graph and H be any non-trivial graph. Then $G \circ H$ has a clique dominating set S if and only if G is complete and $S = V(G)$.*

Corollary 2.8. [2] *Let G be a complete nontrivial graph and H be any graph.Then*

$$\gamma_{cl}(G \circ H) = |V(G)|.$$

Theorem 2.9. [2] *Let G and H be connected nontrivial graphs such that G has a clique dominating set. A subset $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique dominating set of $G[H]$ if and only if S is a clique dominating set of G such that*

(*i*) *$\langle T_x \rangle$ is a clique in H for each $x \in S$ and*

(*ii*) *T_x is a dominating set of H whenever $S = \{x\}$.*

Corollary 2.10. [2] *Let G and H be connected nontrivial graphs such that G has a clique dominating set.Then*

$$\gamma_{cl}(G[H]) = \begin{cases} 1, & if\ \gamma(G) = \gamma(H) = 1 \\ 2, & if\ \gamma(G) = 1\, and\ \gamma(H) \neq 1 \\ \gamma_{cl}(G), & if \gamma(G) \neq 1 \end{cases}$$

3. Main Results

This section presents the clique domination number and the forcing clique domination number of special graphs such as paths, cycles, complete graphs and other special graphs such as generalized wheels, generalized fans, and complete bipartite graphs. In addition, the forcing clique domination number is determined for graphs obtained through some binary operations such as the join, corona, and lexicographic product of two graphs.

Theorem 3.1. *Let G be a connected graph such that G has a clique dominating set. Then*

(i) $f\gamma_{cl}(G) = 0$ *if and only if G contains a unique γ_{cl}-set.*

(ii) $f\gamma_{cl}(G) = 1$ *if and only if G has no unique γ_{cl}-sets and there exists a vertex $t \in V(G)$ which is contained in exactly one γ_{cl}-set of G.*

(iii) *for any integer a such that $1 < a < \gamma_{cl}(G)$, $f\gamma_{cl}(G) = a$ if and only if G has no unique γ_{cl}-sets and a is the minimum number of vertices which are contained in exactly one γ_{cl}-set of G.*

Proof: (i) Suppose that $f\gamma_{cl}(G) = 0$. It follows that $\varnothing$ is the forcing subset for a γ_{cl}-set in G. Suppose that G has two γ_{cl}-sets, say C and D. Then $\varnothing$ is a forcing subset for C and D, a contradiction since a forcing subset must be contained in a unique γ_{cl}-set. Therefore, G contains a unique γ_{cl}-set. Conversely, if G contains a unique γ_{cl}-set, say B. Clearly, $\varnothing$ is a forcing subset of B. Hence, $|\varnothing| = 0 = f\gamma_{cl}(B) = f\gamma_{cl}(G)$.
(ii) Suppose that $f\gamma_{cl}(G) = 1$. By part (i), G has no unique γ_{cl}-set and there exist γ_{cl}-set, say T, and $t \in T$ such that $\{t\}$ is a forcing subset for T and $f\gamma_{cl}(T) = |\{t\}| = 1$, that is, $\{t\}$ is not forcing subset for another γ_{cl}-set of G. Thus, there exists a vertex $t \in V(G)$ which is contained in exactly one γ_{cl}-set of G. Conversely, if G has no unique γ_{cl}-sets , then by part (i), $f\gamma_{cl}(G) \geq 1$. By assumption, there exists a vertex, say c, which is contained in exactly one γ_{cl}-set of G, say C, that is, $\{c\}$ is a forcing subset for C. Therefore, $f\gamma_{cl}(C) = |\{c\}| = 1 = f\gamma_{cl}(G)$.
(iii) Suppose that $f\gamma_{cl}(G) = a$ for any integer a such that $1 < a < \gamma_{cl}(G)$. By part (i), G has no unique γ_{cl}-sets and there exists a unique γ_{cl}-set, say T, and $|S| = a$ such that S is a forcing subset for T and $f\gamma_{cl}(G) = a = |S| = f\gamma_{cl}(T)$. Hence, a is the minimum number of vertices which are contained in exactly one γ_{cl}-set of G. Conversely, if G has no unique γ_{cl}-sets, then by part (i), $f\gamma_{cl}(G) \geq 1$. By assumption, there exists a set S such that $|S| = a > 1, S$ is contained in exactly one γ_{cl}-set of G, say C, that is, S is a forcing subset for C. By the minimality of a, $a = |S| = f\gamma_{cl}(C) = f\gamma_{cl}(G)$. □

The next two results are direct consequences of Theorem 3.1 and definition of forcing clique domination.

Corollary 3.2. *Let G be a connected graph such that G has a clique dominating set. Then*

$$0 \leq f\gamma_{cl}(G) \leq \gamma_{cl}(G).$$

Theorem 3.3. *Let G be a connected graph such that G has a clique dominating set. Then $f\gamma_{cl}(G) = \gamma_{cl}(G)$ if and only if for every γ_{cl}-set of C of G and for each vertex $t \in C$, there exist a vertex $u \in V(G)\backslash C$ such that $\{u\} \cup [C\backslash\{t\}]$ is a γ_{cl}-set of G.*

Proof: Suppose that $f\gamma_{cl}(G) = \gamma_{cl}(G)$. Let C be a γ_{cl}-set of G such that $f\gamma_{cl}(G) = |C| = \gamma_{cl}(G)$, that is, C is the only forcing subset for C. Let $t \in C$. Since $C\backslash\{t\}$ is not a forcing subset for C, there exists a $u \in V(G)\backslash C$ such that$\{u\} \cup [C\backslash\{t\}]$ is a γ_{cl}-set of G. Conversely, suppose that everyγ_{cl}-set C' of G satisfies the given condition. Let C be a γ_{cl}-set of G such that $f\gamma_{cl}(G) = f\gamma_{cl}(C)$ and $|C| = \gamma_{cl}(G)$. Moreover, suppose that C has a forcing subset D with $|D| < |C|$, that is ,$C = D \cup A$, where $A = \{t \in C : t \notin D\}$. Pick $t \in A$. By assumption, there exists $u \in V(G)\backslash C$ such that $\{u\} \cup [C\backslash\{t\}] = B$ is a γ_{cl}-set of G. Thus, $B = D \cup E$, where $E = \{u\} \cup [A\backslash\{t\}]$, that is, B is a γ_{cl}-set containing D , a contradiction. Thus, $|D| = |C|$ and $|C|$ is the only forcing subset for $|C|$. Therefore, $f\gamma_{cl}(G) = f\gamma_{cl}(C) = |C| = \gamma_{cl}(G)$. □

The next result is a restatement of Theorem 3.3.

Remark 3.4. *Let G be a connected graph such that G has a clique dominating set. Then $f\gamma_{cl}(G) = \gamma_{cl}(G)$ if and only if every vertex in a γ_{cl}-set C of G can be replaced by another vertex in $V(G)\backslash C$ to form another γ_{cl}-set of G.*

Proposition 3.5. *Let n be a positive integer with $n \geq 1$. Then the clique domination number of a path P_n and its forcing clique domination number are given by*

$$\gamma_{cl}(P_n) = \begin{cases} 1, & n < 4 \\ 2, & n = 4 \end{cases}$$

and

$$f\gamma_{cl}(P_n) = \begin{cases} 0, & n = 1, 3, 4 \\ 1, & n = 2 \end{cases}$$

. For $n \geq 5$, the path P_n is non$-\gamma_{cl}-$graph, and both $\gamma_{cl}(P_n)$ and $f\gamma_{cl}(P_n)$ are undefined.

Proof: Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Consider the following cases:

Case 1. Let $n = 1$. Clearly, $\{u_1\}$ is the only minimum clique dominating set of P_1. Thus, $\gamma_{cl}(P_1) = 1$ and $f\gamma_{cl}(P_1) = 0$ by Theorem 3.1 (i).

Case 2. Let $n = 2$. By Proposition 2.1, $\gamma(P_2) = \lceil \frac{2}{3} \rceil = 1$ and by Theorem 2.3, $\gamma_{cl}(P_2) = 1$. Clearly, $S_1 = \{u_1\}$ and $S_2 = \{u_2\}$ are the γ_{cl}-sets of P_2, that is, the vertex u_1 is contained in S_1 only. Thus, $f\gamma_{cl}(P_2) = 1$ by Theorem 3.1 (ii).

Case 3. Let $n = 3$. By Proposition 2.1, $\gamma(P_3) = \lceil \frac{3}{3} \rceil = 1$ and by Theorem 2.3, $\gamma_{cl}(P_3) = 1$. Clearly, $\{u_2\}$ is the only γ_{cl}-set of P_3. Thus, $f\gamma_{cl}(P_3) = 1$ by Theorem 3.1 (ii).

Case 4. Let $n = 4$. By Proposition 2.1, $\gamma(P_4) = \lceil \frac{4}{3} \rceil = 2$ and by Theorem 2.3, $\gamma_{cl}(P_4) > 1$. Clearly, $C = \{u_2, u_3\}$ is the only γ_{cl}-sets of P_4 since the induced subgraph $\langle C \rangle$ of C is complete. Thus,$\gamma_{cl}(P_4) = 2$ and $f\gamma_{cl}(P_4) = 0$ by Theorem 3.1 (i).

Case 5. Let $n \geq 5$. Then P_n has induced P_5. By Theorem 2.6, P_n has no clique dominating set. Therefore, for all $n \geq 5$, P_n is $non - \gamma_{cl} - graph$, and both $\gamma_{cl}(P_n)$ and $f\gamma_{cl}(P_n)$ are undefined. □

Proposition 3.6. *Let n be a positive integer with $n \geq 3$. Then the clique domination number and forcing clique domination number of a cycle C_n are equal and given by*

$$f\gamma_{cl}(C_n) = \gamma_{cl}(C_n) = \begin{cases} 1, & n = 3 \\ 2, & n = 4 \end{cases}$$

For $n \geq 5$, the cycle C_n is $non - \gamma_{cl} - graph$, and both $\gamma_{cl}(C_n)$ and $f\gamma_{cl}(C_n)$ are undefined.

Proof: Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Consider the following cases:

Case 1. Let $n = 3$. Then by Proposition 2.1, $\gamma(C_3) = \lceil \frac{3}{3} \rceil = 1$ and by Theorem 2.3, $\gamma_{cl}(C_3) = 1$. Clearly, $S_1 = \{u_1\}$, $S_2 = \{u_2\}$ and $S_3 = \{u_3\}$ are the γ_{cl}-sets of C_3, that is, the vertex u_1 is contained in S_1 only. By Theorem 3.1(ii), $f\gamma_{cl}(C_3) = 1$.

Case 2. Let $n = 4$. By Proposition 2.1, $\gamma(C_4) = \lceil \frac{4}{3} \rceil = 2$. Clearly, $T_1 = \{u_1, u_2\}$, $T_2 = \{u_2, u_3\}$, $T_3 = \{u_3, u_4\}$ and $T_4 = \{u_4, u_1\}$ are the γ_{cl}-sets of C_4, such that for all $i = 1, 2, 3, 4$, the induced subgraph $\langle T_i \rangle$ of T_i is complete. Thus, $\gamma_{cl}(C_4) = 2$. Clearly, every vertex in γ_{cl}-set T_k of C_4 can be replaced by another vertex in $V(C_4) \backslash T_k$ to form another γ_{cl}-set T_j such that $k \neq j$. By Remark 3.4, $f\gamma_{cl}(C_4) = \gamma_{cl}(C_4) = 2$.

Case 3. Let $n \geq 5$. Then C_n has induced P_5. By Theorem 2.6, C_n has no clique dominating set. Therefore, for all $n \geq 5$, C_n is $non - \gamma_{cl} - graph$, and both $\gamma_{cl}(C_n)$ and $f\gamma_{cl}(C_n)$ are undefined. □

Proposition 3.7. *Let n be a positive integer with $n \geq 1$. Then the clique domination number of the complete graph K_n is given by $\gamma_{cl}(K_n) = 1$ and forcing clique domination number is given by*

$$f\gamma_{cl}(K_n) = \begin{cases} 0, & n = 1 \\ 1, & n \geq 2. \end{cases}$$

Proof: Let $V(K_n) = \{u_1, u_2, u_3 \ldots, u_n\}$. By Proposition 2.2, $\gamma(K_n) = 1$ and by Theorem 2.3, $\gamma_{cl}(K_n) = 1$. If $n = 1$, then $\{u_1\}$ is the only γ_{cl}-set of K_1. Thus, $f\gamma_{cl}(K_1) = 0$ by Theorem 3.1(i). Suppose that $n \geq 2$. Then for all $i = 1, 2, \ldots, n$, $S_i = \{u_i\}$ is a γ_{cl}-set of K_n, that is, the vertex u_i is contained in S_i only. By Theorem 3.1 (ii), $f\gamma_{cl}(K_n) = 1$ for all $n \geq 2$. □

Theorem 3.8. *Let G and H be any graphs. Then*

$$f\gamma_{cl}(G+H) = \begin{cases} 0, & if\, either\, \gamma(G) = 1 < \gamma(H)\, and\, G\, has\, a\, unique\, \gamma - set, \\ & or\, \gamma(H) = 1 < \gamma(G)\, and\, H\, has\, a\, unique\, \gamma - set \\ 1, & if\, either\, \gamma(G) = 1 < \gamma(H)\, and\, G\, has\, no\, unique\, \gamma - set, \\ & or \gamma(H) = 1 < \gamma(G)\, and\, H\, has\, no\, unique\, \gamma - set \\ & or\, \gamma(G) = 1\, and\, \gamma(H) = 1 \\ 2, & if\, \gamma(G) > 1\, and\, \gamma(H) > 1. \end{cases}$$

Proof: Consider the following cases:
Case 1. Suppose that $\gamma(G) = 1 < \gamma(H)$ and G has unique γ-set. By Corollary 2.5, $\gamma_{cl}(G+H) = 1$. Suppose that S is the unique γ-set of G. Then $|S| = 1$, say $S = \{u\}$ for a unique vertex u of $V(G)$ and by Theorem 2.3, S is a γ_{cl}-set of G. By Theorem 2.4, S is the only γ_{cl}-set of $G + H$. By Theorem 3.1 (i), $f\gamma_{cl}(G + H) = 0$. Similarly, $f\gamma_{cl}(G + H) = 0$ if $\gamma(H) = 1 < \gamma(G)$ and H has a unique γ-set.

Case 2. Suppose that $\gamma(G) = 1 < \gamma(H)$ and G has no unique γ-set.
By Corollary 2.5, $\gamma_{cl}(G + H) = 1$. Let S and T be γ-sets of G. Then $|S| = |T| = 1$ and by Theorem 2.3, S and T are γ_{cl}-sets of G. Thus, S and T are γ_{cl}-sets of $G + H$ by Theorem 2.4. Then there exists a vertex u contained in S only. By Theorem 3.1(ii), $f\gamma_{cl}(G+H) = 1$. Similarly, $f\gamma_{cl}(G+H) = 1$ if $\gamma(H) = 1 < \gamma(G)$ and H has no unique γ-set.

Case 3. Suppose that $\gamma(G) = 1$ and $\gamma(H) = 1$.
By Corollary 2.5, $\gamma_{cl}(G + H) = 1$. Let S and R be γ-set of G and H, respectively. By Theorem 2.3, S and R are γ_{cl}-sets of G and H, respectively. Then by Theorem 2.4, S and R are γ_{cl}-sets of $G + H$. Then there exists a vertex u contained in S only. By Theorem 3.1(ii), $f\gamma_{cl}(G + H) = 1$.

Case 4. Suppose that $\gamma(G) > 1$ and $\gamma(H) > 1$.
By Corollary 2.5, $\gamma_{cl}(G+H) = 2$. Consider a γ_{cl}-set $S = \{c, d\}$ of $G+H$, where $c \in V(G)$ and $d \in V(H)$. Pick $x \in V(G)\backslash\{c\}$ and $y \in V(H)\backslash\{d\}$. Then $\{c\} \subseteq S_y = \{c, y\}$ and $\{d\} \subseteq S_x = \{x, d\}$, where S_x and S_y are also γ_{cl}-sets of $G + H$ different from S. Thus, $f\gamma_{cl}(S) = 2$. Now, if $\gamma(G) = 2$, then by Theorem 2.3, $\gamma_{cl}(G) \neq 1$. Thus,$\gamma_{cl}(G) = 2$ or $\gamma_{cl}(G)$ is undefined. Suppose that $\gamma_{cl}(G)$ is undefined. Then the set $T = \{e, f\}$, where $e \in V(G)$ and $f \in V(H)$, is a γ_{cl}-set of $G + H$. By the previous argument, $f\gamma_{cl}(T) = 2$. Suppose that $\gamma_{cl}(G) = 2$. Let $S' = \{g, h\}$ be a γ_{cl}-set of G and by Theorem 2.4, S' is also a γ_{cl}-set of $G + H$. Pick $v \in V(H)$. Then $\{g\} \subseteq S_g = \{g, v\}$ and $\{h\} \subseteq S_h = \{h, v\}$ where S_g and S_h are γ_{cl}-sets of $G + H$ different from S'. Thus, $f\gamma_{cl}(S') = 2$. Similarly, if $\gamma(H) = 2$, then for any γ_{cl}-set S^* of $G+H$, $f\gamma_{cl}(S^*) = 2$. In any case, $f\gamma_{cl}(G+H) = 2$. □

The next result follows from Theorem 3.8 and Corollary 2.5.

Corollary 3.9. *For any graph H, $\gamma_{cl}(K_1+H)=1$ and*

$$f\gamma_{cl}(K_1+H)=\begin{cases}0, & \gamma(H)>1,\\ 1, & \gamma(H)=1.\end{cases}$$

The next results are direct consequences of Theorem 3.8, and Corollaries 2.5 and 3.9.

Corollary 3.10. *Let n and m be positive integers. For a complete bipartite graph $K_{n,m}=\overline{K}_n+\overline{K}_m$ where $n\geq 1$ and $m\geq 1$,*

$$\gamma_{cl}(K_{n,m})=\begin{cases}1, & if\, either\, n=1\, or\, m=1,\\ 2, & if\, n\geq 2\, and\, m\geq 2.\end{cases}$$

and

$$f\gamma_{cl}(K_{n,m})=\begin{cases}0, & if\, n=1\, and\, m\geq 2\, or\, m=1\, and\, n\geq 2,\\ 1, & if\, n=1\, and\, m=1\,,\\ 2, & if\, n\geq 2\, and\, m\geq 2.\end{cases}$$

Corollary 3.11. *For the generalized fan $F_{n,m}=\overline{K}_n+P_m$, where $n\geq 1$ and $m\geq 2$,*

$$\gamma_{cl}(F_{n,m})=\begin{cases}1, & if\, either\, n=1\, or\, m<4,\\ 2, & if\, n\geq 2\, and\, m\geq 4.\end{cases}$$

and

$$f\gamma_{cl}(F_{n,m})=\begin{cases}0, & if\, either\, n=1\, and\, m\geq 4\, or\, n\geq 2\, and\, m=3,\\ 1, & if\, either\, n=1\, and\, m<4\, or\, n\geq 2\, and\, m=2,\\ 2, & if\, n\geq 2\, and\, m\geq 4.\end{cases}$$

Corollary 3.12. *For the generalized wheel $W_{n,m}=\overline{K}_n+C_m$, where $n\geq 1$ and $m\geq 3$,*

$$\gamma_{cl}(W_{n,m})=\begin{cases}1, & if\, either\, n=1\, or\, m=3,\\ 2, & if\, n\geq 2\, and\, m\geq 4.\end{cases}$$

and

$$f\gamma_{cl}(W_{n,m})=\begin{cases}0, & if\, n=1\, and\, m\geq 4,\\ 1, & if\, m=3,\\ 2, & if\, n\geq 2\, and\, m\geq 4.\end{cases}$$

Theorem 3.13. *Let G be a trivial graph and H be any graph. Then S is a γ_{cl}-set of $G\circ H$ if and only if $S=V(G)$ or S is a γ-set of H such that $\gamma(H)=1$. In particular, $\gamma_{cl}(G\circ H)=1$.*

Proof: Since G is trivial and $G\circ H=K_1+H$, by Corollary 3.9, $\gamma_{cl}(C\circ H)=1$. Suppose that S is a γ_{cl}-set of $G\circ H$. Since G is trivial, $S=V(G)$ since $V(G)$ is a dominating set of $G\circ H$ and $V(G)$ is complete. Suppose that $\gamma(H)=1$. By Theorem 2.3, $\gamma_{cl}(H)=1$. Then there exists a vertex v in H such that v is adjacent to every vertex in $H\setminus\{v\}$ and to a vertex in G. Take $S=\{v\}$ and so, S is a γ-set of H. The converse is clear. □

Theorem 3.14. *Let G be a complete graph and H be any graph. Then*

$$f\gamma_{cl}(G \circ H) = \begin{cases} 0, & if\, either\, G\, is\, nontrivial\, or\, G\, is\, trivial\, and\, \gamma(H) > 1, \\ 1, & if\, G\, is\, trivial\, and\, \gamma(H) = 1. \end{cases}$$

Proof: Note that by Corollary 2.8, $\gamma_{cl}(G \circ H) = |V(G)|$. Since G is complete, $V(G)$ is a γ_{cl}-set of $G \circ H$. Let S be a γ_{cl}-set of $G \circ H$. Consider the following cases:

Case 1. Suppose that G is nontrivial. By Theorem 2.7, $S = V(G)$ is the only γ_{cl}-set of $G \circ H$. By Theorem 3.1 (i), $f\gamma_{cl}(G \circ H) = 0$.

Case 2. Suppose that G is trivial and $\gamma(H) > 1$. By Theorem 3.13, $S = V(G)$ is the only γ_{cl}-set of $G \circ H$. By Theorem 3.1 (i), $f\gamma_{cl}(G \circ H) = 0$.

Case 3. Suppose that G is trivial and $\gamma(H) = 1$. By Theorem 3.13, $\gamma_{cl}(G \circ H) = 1$ and either $S = V(G)$ or S is the γ-set of H such that S is also γ_{cl}-set of $G \circ H$ and $|S| = 1$. Thus, $G \circ H$ has no unique γ_{cl}-sets. Then there exists a vertex u contained in S only. By Theorem 3.1 (ii), $f\gamma_{cl}(G \circ H) = |S| = 1$. □

Theorem 3.15. *Let G and H be connected nontrivial graphs such that G has a clique dominating set. Then*

$$f\gamma_{cl}(G[H]) = \begin{cases} 0, & if\, \gamma(G) = \gamma(H) = 1\, and \\ & both\, G\, and\, H\, have\, unique\, \gamma - sets, \\ 1, & if\, \gamma(G) = \gamma(H) = 1\, and \\ & either\, G\, or\, H\, has\, no\, unique\, \gamma - sets\, or\, both, \\ 2, & if\, \gamma(G) = 1\, and\, \gamma(H) \neq 1. \\ \gamma_{cl}(G), & if\, \gamma(G) > 1, \end{cases}$$

Proof: Consider the following cases:

Case 1. Suppose that $\gamma(G) = \gamma(H) = 1$ and both G and H have unique γ-sets, say $S = \{x\}$ and $T = \{a\}$, respectively. By Corollary 2.10, $\gamma_{cl}(G[H]) = 1$ and by Theorem 2.3, S and T are also γ_{cl}-sets of G and H, respectively. By Theorem 2.9, $C = S \times T_x = \{(x, a)\}$ is the only γ_{cl}-set of $G[H]$. By Theorem 3.1 (i), $f\gamma_{cl}(G[H]) = 0$.

Case 2. Suppose that $\gamma(G) = \gamma(H) = 1$ and either G or H has no unique γ-sets or both. By Corollary 2.10, $\gamma_{cl}(G[H]) = 1$. WLOG, suppose that G has no unique γ-sets, say $S_1 = \{x\}$ and $S_2 = \{y\}$, and also suppose that H has a γ-set, say $T = \{a\}$. By Theorem 2.3, S_1 and S_2 are also γ_{cl}-sets of G and T is a γ_{cl}-sets of H. By Corollary 2.10, $\gamma_{cl}(G[H]) = 1$. By Theorem 2.9, $C_1 = \bigcup_{x \in S_1} [\{x\} \times T_x]$ and $C_2 = \bigcup_{y \in S_2} [\{y\} \times T_y]$, where $S_1, S_2 \subseteq V(G)$ and $T_x, T_y \subseteq V(H)$ for $x \in S_1$ and $y \in S_2$ such that $|C_1| = |C_2| = 1$ and set $T_x = T_y = \{a\}$, that is, $C_1 = \{(x, a)\}$ and $C_2 = \{(y, a)\}$ are the γ_{cl}-sets of $G[H]$. Clearly, the vertex (x, a) is contained in C_1 only. By Theorem 3.1 (ii), $f\gamma_{cl}(G[H]) = 1$. Similarly, if H has no unique γ-sets or both G and H have no unique γ-sets, $f\gamma_{cl}(G[H]) = 1$.

Case 3. Suppose that $\gamma(G) = 1$ and $\gamma(H) \neq 1$.
By Corollary 2.10, $\gamma_{cl}(G[H]) = 2$. Let $S = \{x, y\}$ be a clique dominating set of G such that $xy \in E(G)$. Choose any vertex $a \in V(H)$. Then $C = \{(x, a), (y, a)\}$ is a γ_{cl}-set of $G[H]$ by Theorem 2.9 and Corollary 2.10. Choose $c \in V(H)\backslash\{a\}$. It follows that $\{(x, a)\} \subseteq C_1 = \{(x, a), (y, c)\}$ and $\{(y, a)\} \subseteq C_2 = \{(x, c), (y, a)\}$, where C_1 and C_2 are also γ_{cl}-sets of $G[H]$ different from C. It follows that $f\gamma_{cl}(C) = 2 = f\gamma_{cl}(G[H])$.

Case 4. Suppose that $\gamma(G) > 1$.
By Corollary 2.10, $\gamma_{cl}(G[H]) = \gamma_{cl}(G)$. Let $C = \bigcup_{x\in S} [\{x\} \times T_x]$ be a γ_{cl}-set of $G[H]$ and let $F_C = \bigcup_{x\in D} [\{x\} \times F_x]$ be a forcing subset for C. Suppose that S is a γ_{cl}-set of G. Then $|C| = |S|$ and so, $|T_x| = 1$ for all $x \in S$. Hence, $F_x = T_x$ for all $x \in D$. If $D \neq S$, say $y \in S\backslash D$, then $F_C \subseteq C' = \bigcup_{x\in S} [\{x\} \times T'_x]$, where $T'_x = T_x$ for $x \in S\backslash\{y\}$ and T'_y is a singleton subset of H different from T_y. Since C' is a γ_{cl}-set of $G[H]$ and $C' \neq C$, F_C is not a forcing subset for C, contrary to the assumption. Thus, $D = S$, that is, $F_C = C$. Hence, $f\gamma_{cl}(C) = |C| = \gamma_{cl}(G) = f\gamma_{cl}(G[H])$. □

The next result follows from Theorem 3.15.

Corollary 3.16. *Let H be a connected nontrivial graph. Then for any complete nontrivial graph K_n,*

$$f\gamma_{cl}(K_n[H]) = \begin{cases} 1, & if\, \gamma(H) = 1, \\ 2, & if\, \gamma(H) \neq 1. \end{cases}$$

Corollary 3.17. *Let G and H be connected nontrivial graphs. Then for any complete nontrivial graph K_n,*

$$f\gamma_{cl}((K_n \circ G)[H]) = n.$$

Proof: By Corollary 2.8, $K_n \circ G$ has a minimum clique dominating set and $\gamma_{cl}(K_n \circ G) = |V(K_n)| = n$ such that $n > 1$ since K_n is nontrivial. By Theorem 2.3, $\gamma(K_n \circ G) > 1$. By Theorem 3.15 and Corollary 2.8,

$$f\gamma_{cl}((K_n \circ G)[H]) = \gamma_{cl}(K_n \circ G) = n.$$

□

4. Conclusion

In this study, the idea of forcing clique domination in graphs was examined along with its basic characteristics. We investigated how the forcing clique domination number and the clique domination number relate to one another.

A significant result in our study when the forcing clique domination number is zero. This happens when each minimum clique dominating set is uniquely determined. These graphs are especially helpful in applications requiring stable and non-redundant control because of their structural rigidity in clique domination properties.

Another important result is when the forcing clique domination number is equal to the clique domination number. This implies that every vertex in a minimum clique dominating set can be replaced by another vertex in the graph while still maintaining the property of clique domination. This feature is important in fault-tolerant network topologies since it will allow other nodes to assume dominance responsibilities without affecting connection or coverage.

Also, we also discovered graphs for which the clique domination number is undefined, as they do not have a clique dominating set, making the forcing clique domination number itself undefined.

Our research sheds more light on the characteristics of forcing clique domination and its function in graph theory. Future studies might concentrate on determining the forcing clique domination number of other binary operations not mentioned in this study and investigating real-world applications in social influence modeling, biological networks, and network security.

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