EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6070
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Overlapping Domain Decomposition Methods for Noncoercive Hamilton-Jacobi-Bellman Equation

Samar Chebbah[1,2,*], Sofiane Madi[3], Mohamed Haiour[4]

[1] *Department of Mathematics, Institute of Mathematics and Computer Science, Abdelhafid Boussouf University Center–Mila, Mila, Algeria*
[2] *Laboratoire des Mathématiques Appliquées et Didactique (MAD), École Normale Supérieure El Katiba Assia Djebar (ENSC) – Constantine, Constantine, Algeria*
[3] *Department of Mathematics, Faculty of Mathematics and Computer Science, University of Batna 2, Batna, Algeria*
[4] *Department of Mathematics, Faculty of the Sciences, University Badji Mokhtar–Annaba, P.O. 23000 Annaba, Algeria*

Abstract. This study presents an approach to demonstrate the monotonic and geometric convergence of the non-coercive Hamilton-Jacobi-Bellman problem with Dirichlet boundary conditions using the finite difference method. The approach combines overlapping domain decomposition with a sequence of lower and upper solutions to characterize the solution in a discrete setting. Numerical experiments are conducted to validate the methodology and illustrate the consistency of the theoretical findings.

2020 Mathematics Subject Classifications: 65N06, 65N12, 49L25, 65N55

Key Words and Phrases: Hamilton-Jacobi-Bellman equation, noncoercive problems, overlapping domain decomposition, convergence analysis, lower and upper solutions, finite difference method

---

1. Introduction

We consider the following non-coercive Hamilton-Jacobi-Bellman (HJB) equation, with Dirichlet boundary conditions :

$$\begin{cases} \max\limits_{1\leq \mathfrak{i}\leq \mathfrak{n}} (\mathcal{A}^{\mathfrak{i}}\xi - \mathcal{F}^{\mathfrak{i}}) = 0, & \text{in}\, \Omega, \\ \xi = 0, & \text{on}\, \partial\Omega, \end{cases} \tag{1}$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6070

Email addresses: s.chebbah@centre-univ-mila.dz (S. Chebbah), s.madi@univ-batna2.dz (S.Madi), mohamed.haiour@univ-annaba.dz (M. Haiour)

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. The operators $\mathcal{A}^i$ are uniformly elliptic and take the form:

$$\mathcal{A}^{\mathfrak{i}} = \sum_{\mathfrak{t},\mathfrak{s}=1}^{d} a^{\mathfrak{i}}_{\mathfrak{ts}}(x)\frac{\partial^2}{\partial x_{\mathfrak{t}}\partial x_{\mathfrak{s}}} + \sum_{\mathfrak{s}=1}^{d} b^{\mathfrak{i}}_{\mathfrak{s}}(x)\frac{\partial}{\partial x_{\mathfrak{s}}} + a^{\mathfrak{i}}_0(x),$$

where the coefficients $a^{\mathfrak{i}}_{\mathfrak{ts}}(x), b^{\mathfrak{i}}_{\mathfrak{s}}(x), a^{\mathfrak{i}}_0(x) \in C^2(\overline{\Omega})$ satisfy the following conditions:

$$\sum_{\mathfrak{t},\mathfrak{s}=1}^{d} a^{\mathfrak{i}}_{\mathfrak{ts}}(x)\nu_{\mathfrak{t}}\nu_{\mathfrak{s}} \geq \delta|\nu|^2, \quad \forall \nu \in \mathbb{R}^d, \quad \delta > 0,$$

$$a^{\mathfrak{i}}_0(x) \geq \gamma > 0.$$

The noncoercive bilinear form for $\mathfrak{u}, \mathfrak{v} \in H^1(\Omega)$ is expressed as follows:

$$\mathfrak{a}^{\mathfrak{i}}\langle \mathfrak{u}, \mathfrak{v}\rangle = \int_{\Omega} \left(\sum_{\mathfrak{t},\mathfrak{s}=1}^{d} a^{\mathfrak{i}}_{\mathfrak{ts}}(x)\frac{\partial \mathfrak{u}}{\partial x_{\mathfrak{t}}}\frac{\partial \mathfrak{v}}{\partial x_{\mathfrak{s}}} + \sum_{\mathfrak{s}=1}^{d} b^{\mathfrak{i}}_{\mathfrak{s}}(x)\frac{\partial \mathfrak{u}}{\partial x_{\mathfrak{s}}}\mathfrak{v} + a^{\mathfrak{i}}_0(x)\mathfrak{u}\mathfrak{v} \right) dx.$$

Additionally, the function $\mathcal{F}^{\mathfrak{i}}$ is assumed to be smooth and nonnegative:

$$\mathcal{F}^{\mathfrak{i}} \in C^2(\overline{\Omega}), \quad \mathcal{F}^{\mathfrak{i}} \geq 0.$$

This study examines the monotonic and geometric convergence of problem (1) within a discrete framework using the finite difference method. We employ the overlapping domain decomposition approach along with the characterization of sequences of lower and upper solutions. This approach allows us to analyze the convergence properties of the Hamilton-Jacobi-Bellman (HJB) equation, particularly in its non-coercive form. The absence of a coercive condition in the HJB equation presents a significant challenge, making the application of conventional solution methods more complex.

The numerical approximation of Hamilton-Jacobi-Bellman (HJB) equations has been an active research area, with various methods proposed to enhance accuracy and computational efficiency. For instance, the study in [1] explores numerical schemes for systems of HJB equations in innovation dynamics, providing insights into their stability and convergence. Mixed finite element techniques have also been developed to handle HJB equations with Cordes coefficients, as discussed in [2]. To address challenges associated with quasi-variational inequalities, domain decomposition strategies such as the Schwarz method have been analyzed in [3]. Furthermore, overlapping domain decomposition approaches have been applied to non-coercive quasi-variational systems related to HJB equations [4], while [5] introduces a contraction-based algorithmic approach to solving these equations. In this work, we adopt an overlapping domain decomposition method that ensures stability through lower and upper solutions, providing a robust numerical framework for solving HJB equations efficiently.

In this work,we adopt an overlapping domain decomposition approach, leveraging the interaction between lower and upper solutions to ensure numerical stability and convergence. This makes it a reliable method for solving complex problems. This paper is organized as follows:
Section 2 provides a detailed analysis of the transformation of the non-coercive problem (1) into the coercive Hamilton-Jacobi-Bellman (HJB) equation, considering both the continuous and discrete cases using the finite difference method. Section 3 establishes the monotonic and geometric convergence of the solutions by applying the overlapping domain decomposition method along with a sequence of lower and upper solutions. The final section presents numerical experiments that confirm the theoretical findings, illustrating the effectiveness of domain decomposition methods in solving HJB equations and evaluating the impact of overlapping on the convergence rate.

2. Coercive Reformulation of the HJB Equation

In this section, we reformulate the Hamilton-Jacobi-Bellman (HJB) equation into a coercive form to improve its stability and numerical solvability. Following [6], this is achieved by introducing a regularization parameter $\mu > 0$, which ensures coercivity and enhances the well-posedness of the problem. The modified equation is given by:

$$\begin{cases} \max\limits_{1\leq \mathfrak{i}\leq \mathfrak{n}} (\mathcal{B}^{\mathfrak{i}}\xi - \mathcal{G}^{\mathfrak{i}}(\xi)) = 0, & \text{in } \Omega, \\ \xi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where the coercive operators $\mathcal{B}^1, \cdots, \mathcal{B}^{\mathfrak{n}}$ are defined as:

$$\mathcal{B}^{\mathfrak{i}} = \sum_{\mathfrak{t},\mathfrak{s}=1}^{d} a^{\mathfrak{i}}_{\mathfrak{t}\mathfrak{s}}(x)\frac{\partial^2}{\partial x_{\mathfrak{t}}\partial x_{\mathfrak{s}}} + \sum_{\mathfrak{s}=1}^{d} b^{\mathfrak{i}}_{\mathfrak{s}}(x)\frac{\partial}{\partial x_{\mathfrak{s}}} + (a^{\mathfrak{i}}_0(x) + \mu),$$

and the corresponding equations are defined as:

$$\mathcal{B}^{\mathfrak{i}} = \mathcal{A}^{\mathfrak{i}} + \mu I \quad \text{and} \quad \mathcal{G}^{\mathfrak{i}}(\xi) = \mathcal{F}^{\mathfrak{i}} + \mu\xi, \quad \mathfrak{i} = \overline{1, \mathfrak{n}}.$$

The bilinear form for the operator $\mathcal{B}^{\mathfrak{i}}$ is expressed as:

$$\mathfrak{b}^{\mathfrak{i}}\langle \mathfrak{u}, \mathfrak{v}\rangle = \mathfrak{a}^{\mathfrak{i}}\langle \mathfrak{u}, \mathfrak{v}\rangle + \mu\langle \mathfrak{u}, \mathfrak{v}\rangle,$$

Previous studies have established the existence, uniqueness, and regularity of the solution to the continuous problem (2) (See [7, 8]). To solve this problem numerically, both finite element and finite difference methods have been employed, transforming the continuous problem into a system of algebraic equations through discretization. This approach simplifies the problem and enhances its computational feasibility.

Thus, the discrete version of the problem is formulated as:

$$\begin{cases} \max\limits_{1 \le \mathfrak{i} \le \mathfrak{n}} (\mathscr{B}^{\mathfrak{i}}\zeta - \mathscr{G}^{\mathfrak{i}}(\zeta)) = 0, & \text{in } \Omega, \\ \zeta = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

here, $\mathscr{B}^{\mathfrak{i}}$ and $\mathscr{G}^{\mathfrak{i}}(\zeta)$ are the discrete approximations of the continuous operators, facilitating the numerical solution of the HJB equation. These are given by:

$$\mathscr{B}^{\mathfrak{i}} = \mathscr{A}^{\mathfrak{i}} + \mu I, \quad \mathscr{G}^{\mathfrak{i}}(\zeta) = \mathscr{F}^{\mathfrak{i}} + \mu\zeta.$$

The matrices $\mathscr{B}^{\mathfrak{i}}$ and $\mathscr{A}^{\mathfrak{i}}$ satisfy the discrete maximum principle and are M-matrices[7], ensuring stability and convergence in the numerical approximation of the HJB equation.

3. Decomposition of the Domain into Overlapping Subdomains

In this section, we introduce an approach that combines domain decomposition with overlapping subdomains and the sequence of lower and upper solutions to efficiently solve the Hamilton-Jacobi-Bellman (HJB) equation. The computational domain is divided into overlapping subdomains, which enhances numerical stability. We establish the geometric and monotonic convergence towards the discrete solution of the problem.

3.1. Definition of Subdomain Decomposition

We examine a regular and bounded domain Ω in $\mathbb{R}^2$, which is partitioned into two subdomains, Ω_1 and Ω_2, that overlap along the x-axis. These subdomains are defined by the following relations:

$$\Omega = \Omega_1 \cup \Omega_2 \quad \text{and} \quad \Omega_1 \cap \Omega_2 \neq \emptyset.$$

The boundaries of each subdomain are denoted by $\Gamma_1 = \partial\Omega_1$ and $\Gamma_2 = \partial\Omega_2$, and the interface where the subdomains meet is represented by:

$$\Sigma_1 = \partial\Omega_1 \cap \Omega_2, \quad \Sigma_2 = \partial\Omega_2 \cap \Omega_1.$$

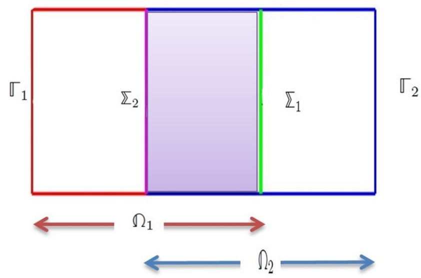

Figure 1: **Illustration of the domain Ω and its subdomains.**

The specific case considered here is where the domain Ω is defined as $[0,1]\times[0,1]$. This domain is then subdivided into two overlapping subdomains:

$$\Omega_1 = [0,\beta]\times[0,1], \quad \Omega_2 = [\alpha,1]\times[0,1],$$

where α and β represent the boundaries that define the partition. The region of overlap between Ω_1 and Ω_2 is given by the interval:

$$\tau = \beta - \alpha,$$

with τ representing the length of the overlap. Additionally, the geometric length of the overlap region is quantified as:

$$\chi(\mathfrak{h}) = \tau \times \mathfrak{h},$$

where $\mathfrak{h}$ is a scaling factor.

3.2. Block Decomposition of the Domain Matrix

We consider the matrices $\mathscr{B}^{\mathfrak{i}}$ of dimension $(d\times d)$, which are associated with vectors that can be partitioned into block form as follows:

$$\zeta = \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}, \quad \mathscr{G}^{\mathfrak{i}}(\zeta) = \begin{pmatrix}\mathscr{G}^{\mathfrak{i}}(\zeta_1)\\ \mathscr{G}^{\mathfrak{i}}(\zeta_2)\end{pmatrix}, \quad \mathscr{B}^{\mathfrak{i}} = \begin{pmatrix}\mathscr{B}^{\mathfrak{i}}_{1,1} & -\mathscr{B}^{\mathfrak{i}}_{1,2}\\ -\mathscr{B}^{\mathfrak{i}}_{2,1} & \mathscr{B}^{\mathfrak{i}}_{2,2}\end{pmatrix}.$$

Here, the vectors ζ_1 and ζ_2 are defined as:

$$\zeta_1 = \left((\zeta_1)_1,\ldots,(\zeta_1)_{\beta-1}\right)^T, \quad \zeta_2 = \left((\zeta_2)_{\alpha+1},\ldots,(\zeta_2)_d\right)^T.$$

To construct the matrix $\mathscr{B}^{\mathfrak{i}}$ using the domain decomposition method, we partition it into two different forms within each subdomain. The decomposition for the first subdomain is defined as follows:

$$\mathscr{B}^{\mathfrak{i}} = \begin{pmatrix}\mathscr{B}^{\mathfrak{i}}_{1,1} & -\mathscr{B}^{\mathfrak{i}}_{1,2}\\ \mathscr{C}^{\mathfrak{i}}_1 & \mathscr{D}^{\mathfrak{i}}_1\end{pmatrix},$$

where:

- $\mathscr{B}^{\mathfrak{i}}_{1,1}$ and $\mathscr{D}^{\mathfrak{i}}_1$ are square matrices of dimensions $(\beta-1)\times(\beta-1)$ and $(d-\beta+1)\times(d-\beta+1)$ respectively.
- $\mathscr{B}^{\mathfrak{i}}_{1,2}$ and $\mathscr{C}^{\mathfrak{i}}_1$ are of dimensions $(\beta-1)\times(d-\beta+1)$ and $(d-\beta+1)\times(\beta-1)$ respectively.

The second partition of the matrix $\mathscr{B}^{\mathfrak{i}}$ is as follows:

$$\mathscr{B}^{\mathfrak{i}} = \begin{pmatrix}\mathscr{D}^{\mathfrak{i}}_2 & \mathscr{C}^{\mathfrak{i}}_2\\ -\mathscr{B}^{\mathfrak{i}}_{2,1} & \mathscr{B}^{\mathfrak{i}}_{2,2}\end{pmatrix},$$

where:

- $\mathscr{B}^{\mathfrak{i}}_{2,2}$ and $\mathscr{D}^{\mathfrak{i}}_{2}$ are square matrices of dimensions $(d-\alpha)\times(d-\alpha)$ and $\alpha\times\alpha$ respectively.

- $\mathscr{B}^{\mathfrak{i}}_{2,1}$ and $\mathscr{C}^{\mathfrak{i}}_{2}$ are of dimensions $(d-\alpha)\times\alpha$ and $\alpha\times(d-\alpha)$ respectively.

When $\beta=\alpha+1$ (i.e., $\tau=1$), the matrices $\mathscr{B}^{\mathfrak{i}}_{1,1}$ and $\mathscr{D}^{\mathfrak{i}}_{2}$ coincide, and similarly, $\mathscr{B}^{\mathfrak{i}}_{2,2}$ and $\mathscr{D}^{\mathfrak{i}}_{1}$ coincide. This results in minimal geometric overlap and complete exclusion of algebraic overlap. Additionally: $\mathscr{B}^{\mathfrak{i}}_{1,1}$ and $\mathscr{B}^{\mathfrak{i}}_{2,2}$ correspond to the discretized operators under homogeneous Dirichlet boundary conditions.
To ensure that the enhanced system remains consistent with the original system despite the overlapping region, the non-diagonal blocks $\mathscr{B}^{\mathfrak{i}}_{2,1}$ and $\mathscr{B}^{\mathfrak{i}}_{1,2}$ are extended by adding rows and columns filled with zeros. The resulting extended matrices are defined as follows:

$$\tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}=\begin{bmatrix}0_{\beta-1,\tau-1} & \mathscr{B}^{\mathfrak{i}}_{1,2}\end{bmatrix},\quad \tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}=\begin{bmatrix}\mathscr{B}^{\mathfrak{i}}_{2,1} & 0_{d-\alpha,\tau-1}\end{bmatrix}.$$

Where:

- $\tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}$ is a matrix of size $(\beta-1)\times(d-\alpha)$ that connects a vector from Ω_2 to Ω_1 with zero entries outside of Ω_2.

- $\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}$ is a matrix of size $(d-\alpha)\times(\beta-1)$ that connects a vector from Ω_1 to Ω_2, with zero entries outside of Ω_1.

The extended system satisfies the following equations:

$$\begin{cases}\max\limits_{1\le\mathfrak{i}\le\mathfrak{n}}\left(\tilde{\mathscr{B}}^{\mathfrak{i}}\zeta-\mathscr{G}^{\mathfrak{i}}(\zeta)\right)=0, & \text{in}\,\Omega,\\ \zeta=0, & \text{on}\,\partial\Omega.\end{cases}$$

In the minimal overlap case ($\tau=1$), the extended system simplifies to the original matrix associated with the domain partition without any overlap. This is expressed by the following relation:

$$\tilde{\mathscr{B}}^{\mathfrak{i}}=\begin{pmatrix}\mathscr{B}^{\mathfrak{i}}_{1,1} & -\tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\\ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1} & \mathscr{B}^{\mathfrak{i}}_{2,2}\end{pmatrix}=\begin{pmatrix}\mathscr{B}^{\mathfrak{i}}_{1,1} & -\mathscr{B}^{\mathfrak{i}}_{1,2}\\ -\mathscr{B}^{\mathfrak{i}}_{2,1} & \mathscr{B}^{\mathfrak{i}}_{2,2}\end{pmatrix}=\mathscr{B}^{\mathfrak{i}}.$$

3.3. Iterative Algorithm for Domain Decomposition

The iterative process begins with selecting an initial solution, either a lower or an upper one, depending on the nature of the problem. We first define the notions of subsolution and supersolution:

- A subsolution $\check{\zeta}\in\mathbb{R}^{\mathfrak{m}}$ satisfies the inequality:

$$\max_{1\le\mathfrak{i}\le\mathfrak{n}}\left\{\mathscr{B}^{\mathfrak{i}}\check{\zeta}-\mathscr{G}^{\mathfrak{i}}(\check{\zeta})\right\}\le 0.$$

- A supersolution $\hat{\zeta} \in \mathbb{R}^{\mathfrak{m}}$ satisfies the inequality:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}\hat{\zeta} - \mathscr{G}^{\mathfrak{i}}(\hat{\zeta}) \right\} \geq 0.$$

Following these definitions, and based on the approach in [9], the initial subsolution is chosen as:

$$\check{\zeta}^0 = (0, 0, \dots, 0), \quad \text{if} \quad \mathscr{G}^{\mathfrak{i}} \geq 0 \quad \text{for all} \quad \mathfrak{i} = 1, \dots, \mathfrak{n}.$$

To compute the corresponding initial supersolution $\hat{\zeta}^0$, we solve the nonlinear system:

$$\mathscr{B}^{\mathfrak{i}}\hat{\zeta}^0 = \mathscr{G}^{\mathfrak{i}}(\hat{\zeta}^0), \quad \text{for each} \quad \mathfrak{i} = 1, \dots, \mathfrak{n}.$$

The discrete Hamilton-Jacobi-Bellman (HJB) problem, as expressed in equation (3) can be reformulated using an overlapping domain decomposition strategy, as follows:

- Solution for the domain Ω_1: To compute ζ_1^{k+1}, we solve the following system:

$$\begin{cases} \max\limits_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left(\mathscr{B}^{\mathfrak{i}}_{1,1}\zeta_1^{k+1} - \mathscr{G}^{\mathfrak{i}}(\zeta_1^k) \right) = 0, & \text{in } \Omega_1, \\ \left(\zeta_1^{k+1}\right)_\beta = \left(\zeta_2^k\right)_\beta, & \text{on } \Sigma_1. \end{cases}$$

- Solution for the domain Ω_2: To compute ζ_2^{k+1}, we solve the following system:

$$\begin{cases} \max\limits_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left(\mathscr{B}^{\mathfrak{i}}_{2,2}\zeta_2^{k+1} - \mathscr{G}^{\mathfrak{i}}(\zeta_2^k) \right) = 0, & \text{in } \Omega_2, \\ \left(\zeta_2^{k+1}\right)_\alpha = \left(\zeta_1^{k+1}\right)_\alpha, & \text{on } \Sigma_2. \end{cases}$$

Upon incorporating boundary conditions, the system becomes:

$$\begin{cases} \max\limits_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\zeta_1^{k+1} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\zeta_2^{k} - \mathscr{G}^{\mathfrak{i}}(\zeta_1^k) \right\} = 0, & \text{in } \Omega_1, \\ \max\limits_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{2,2}\zeta_2^{k+1} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\zeta_1^{k+1} - \mathscr{G}^{\mathfrak{i}}(\zeta_2^k) \right\} = 0, & \text{in } \Omega_2. \end{cases}$$

We reformulate the iterative process derived from the overlapping domain decomposition method into the following matrix-based nonlinear system:

$$\begin{pmatrix} \mathscr{B}^{\mathfrak{i}}_{1,1} & 0 \\ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1} & \mathscr{B}^{\mathfrak{i}}_{2,2} \end{pmatrix} \begin{pmatrix} \zeta_1^{k+1} \\ \zeta_2^{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1^{k} \\ \zeta_2^{k} \end{pmatrix} + \begin{pmatrix} \mathscr{G}^{\mathfrak{i}}(\zeta_1^k) \\ \mathscr{G}^{\mathfrak{i}}(\zeta_2^k) \end{pmatrix}. \tag{4}$$

To better understand this formulation, we recast it in the context of the block Gauss-Seidel method. The general form of this iterative scheme is given by:

$$\begin{cases} \zeta^{(0)} \in \mathbb{R}^{\mathfrak{m}}, \\ \mathscr{B}_{\mathfrak{t},\mathfrak{t}}\, \zeta_{\mathfrak{t}}^{(k+1)} = -\sum\limits_{\mathfrak{s}<\mathfrak{t}}^{\mathfrak{m}} \mathscr{B}_{\mathfrak{t},\mathfrak{s}}\, \zeta_{\mathfrak{s}}^{(k+1)} - \sum\limits_{\mathfrak{s}>\mathfrak{t}}^{\mathfrak{m}} \mathscr{B}_{\mathfrak{t},\mathfrak{s}}\, \zeta_{\mathfrak{s}}^{(k)} + \mathscr{G}_{\mathfrak{t}}, \quad \mathfrak{t} = 1, \dots, \mathfrak{m}. \end{cases} \tag{5}$$

By comparing systems (4) and (5), it becomes clear that they are structurally equivalent in the nonlinear context, where the source term $\mathscr{G}$ is replaced by the nonlinear expression $\mathscr{G}^k$ at each iteration. The algorithm below outlines the iterative steps of the proposed method.

Algorithm Overlapping Domain Decomposition

Initialize: Set the initial guess as $\check{\zeta}^0 = (0, 0, \dots, 0)$, and set $k = 0$.
Step 1: Compute the updated solution $\check{\zeta}_1^{k+1}$ by solving the system:

$$\max_{1\le \mathfrak{i} \le \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}_1^{k+1} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}_2^{k} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_1^{k}) \right\} = 0.$$

Step 2: Calculate the updated value $\check{\zeta}_2^{k+1}$ using the most recent value of $\check{\zeta}_1^{k+1}$:

$$\max_{1\le \mathfrak{i} \le \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}_1^{k+1} + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}_2^{k+1} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_2^{k}) \right\} = 0.$$

Step 3: Verify convergence of the solution. If the solution has not converged, increment k by 1 and repeat from Step 1.

3.4. Monotonic and Geometric Convergence

To validate the proposed algorithm, we establish its monotonic and geometric convergence using lower and upper solutions. The following theorem presents the proof.

Theorem 1. *The iterative process described in Algorithm generates two sequences,* $(\check{\zeta}^k)$ *and* $(\hat{\zeta}^k)$*, which exhibit monotonic behavior. More precisely, the sequence* $(\check{\zeta}^k)$ *increases, while* $(\hat{\zeta}^k)$ *decreases, both converging toward the unique solution* ζ *of the Hamilton-Jacobi-Bellman system (3).*

Proof. The proof is structured into five consecutive steps.

- **Step 1:** We establish that the sequence $(\check{\zeta}^k)$, produced by the algorithm is monotonic and converges to the unique solution of the system (3).
  Initially, for $k = 0$, we solve the problem in the first subdomain to compute $\check{\zeta}_1^1$ as follows:
  $$\max_{1\le \mathfrak{i} \le \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}_1^{1} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}_2^{0} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_1^{0}) \right\} = 0.$$
  Since $\check{\zeta}^0$ is a lower solution, the following inequality holds:
  $$\max_{1\le \mathfrak{i} \le \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}_1^{0} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}_2^{0} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_1^{0}) \right\} \le 0.$$
  From this, we deduce that:
  $$\check{\zeta}_1^0 \le \check{\zeta}_1^1.$$
  Next, we proceed to solve the problem in the second subdomain to determine $\check{\zeta}_2^1$:
  $$\max_{1\le \mathfrak{i} \le \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}_1^{1} + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}_2^{1} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_2^{0}) \right\} = 0. \tag{6}$$

Since $\check{\zeta}^0$ is also a lower solution, we obtain:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}^0_1 + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}^0_2 - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^0_2) \right\} \leq 0. \tag{7}$$

Comparing equations (6) and (7), we deduce:

$$\check{\zeta}^0_2 \leq \check{\zeta}^1_2.$$

Thus, we conclude that:

$$\check{\zeta}^0 \leq \check{\zeta}^1.$$

This confirms that $\check{\zeta}^1$ is also a lower solution.
By mathematical induction, assuming that $\check{\zeta}^k \leq \check{\zeta}^{k+1}$ holds for some k, we extend this property to all iterations:

$$\check{\zeta}^0 \leq \check{\zeta}^1 \leq \cdots \leq \check{\zeta}^k \leq \check{\zeta}^{k+1}.$$

Hence, the sequence $(\check{\zeta}^k)$ is non-decreasing.

- **Step 2:** Establishing the convergence of the sequence $(\check{\zeta}^k)$ to the solution of system (3).
Since $\check{\zeta}^k$ is considered a lower solution, the following inequality holds:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}\check{\zeta}^k - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^k) \right\} \leq 0. \tag{8}$$

This directly implies the relation:

$$\mathscr{B}^{\mathfrak{i}}\check{\zeta}^k - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^k) \leq 0 \quad \Rightarrow \quad \mathscr{B}^{\mathfrak{i}}\check{\zeta}^k - \mathscr{F}^{\mathfrak{i}} - \mu\check{\zeta}^k \leq 0. \tag{9}$$

Now, let us assume the existence of a solution $\check{\zeta}^*$ satisfying:

$$\mathscr{B}^{\mathfrak{i}}\check{\zeta}^* - \mathscr{F}^{\mathfrak{i}} - \mu\check{\zeta}^* = 0. \tag{10}$$

From equation (10), we express:

$$\mathscr{F}^{\mathfrak{i}} = \mathscr{B}^{\mathfrak{i}}\check{\zeta}^* - \mu\check{\zeta}^*.$$

By substituting this into equation (9), we obtain:

$$\mathscr{B}^{\mathfrak{i}}\check{\zeta}^k - \mathscr{B}^{\mathfrak{i}}\check{\zeta}^* + \mu\check{\zeta}^* - \mu\check{\zeta}^k \leq 0.$$

Rearranging, we get:

$$(\mathscr{B}^{\mathfrak{i}} - \mu\mathcal{I})(\check{\zeta}^k - \check{\zeta}^*) \leq 0.$$

Since $\mathscr{B}^{\mathfrak{i}}$ can be rewritten as $\mathscr{A}^{\mathfrak{i}} + \mu\mathcal{I}$, it follows that:

$$\mathscr{A}^{\mathfrak{i}}(\check{\zeta}^k - \check{\zeta}^*) \leq 0.$$

Given that $\mathscr{A}^{\mathfrak{i}}$ is an M-matrix, we deduce:

$$\check{\zeta}^k \leq \check{\zeta}^*.$$

Thus, the sequence $(\check{\zeta}^k)$ is bounded below by $\check{\zeta}^*$. Since $(\check{\zeta}^k)$ is a monotonically increasing and lower-bounded sequence, it necessarily converges. Therefore, we conclude:

$$\lim_{k\to\infty} \check{\zeta}^k = \check{\zeta}^*. \tag{11}$$

- **Step 3:** The vector $\check{\zeta}^* = (\check{\zeta}_1^*, \check{\zeta}_2^*)$ corresponds to the solution of system (3).
From the iterative scheme described in algorithm, we derive the following conditions:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}_1^{k+1} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}_2^k - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_1^k) \right\} = 0,$$

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}_1^{k+1} + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}_2^{k+1} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_2^k) \right\} = 0.$$

Taking the limit as $k \to \infty$ and employing equation (11), we obtain the following system:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}_1^{*} - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}_2^* - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_1^*) \right\} = 0,$$

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}_1^{*} + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}_2^{*} - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}_2^*) \right\} = 0.$$

This system can be rewritten in a more compact form as:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}\check{\zeta}^* - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^*) \right\} = 0.$$

Consequently, we establish that $\check{\zeta}^*$ is the unique solution to system (3).

- **Step 4:** Uniqueness of the discrete solution to the Hamilton-Jacobi-Bellman system (3).
Assume that the system (3) admits two possible solutions, denoted by ζ_- and ζ_+. This leads to the following two conditions:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}\zeta_- - \mathscr{G}^{\mathfrak{i}}(\zeta_-) \right\} = 0,$$

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}\zeta_+ - \mathscr{G}^{\mathfrak{i}}(\zeta_+) \right\} = 0.$$

From these relations, it follows that:

$$\mathscr{B}^{\mathfrak{i}}\zeta_+ - \mathscr{G}^{\mathfrak{i}}(\zeta_+) = 0 \quad \Leftrightarrow \quad \mathscr{B}^{\mathfrak{i}}\zeta_+ - \mathscr{F}^{\mathfrak{i}} - \mu\zeta_+ = 0,$$

$$\mathscr{B}^{\mathfrak{i}}\zeta_- - \mathscr{G}^{\mathfrak{i}}(\zeta_-) = 0 \quad \Leftrightarrow \quad \mathscr{B}^{\mathfrak{i}}\zeta_- - \mathscr{F}^{\mathfrak{i}} - \mu\zeta_- = 0.$$

By subtracting these two equations and considering the decomposition $\mathscr{B}^{\mathfrak{i}} = \mathscr{A}^{\mathfrak{i}} + \mu\mathcal{I}$, we obtain:

$$(\mathscr{B}^{\mathfrak{i}} - \mu\mathcal{I})(\zeta_+ - \zeta_-) = 0 \quad \Leftrightarrow \quad \mathscr{A}^{\mathfrak{i}}(\zeta_+ - \zeta_-) = 0.$$

Given that the matrices $\mathscr{A}^{\mathfrak{i}}$ are M-matrices, we conclude that:

$$\zeta_- - \zeta_+ = 0 \quad \Rightarrow \quad \zeta_- = \zeta_+.$$

Thus, the solution to the system (3) is uniquely determined.

- **Step 5:** Demonstrating the monotonicity of the upper sequence $(\hat{\zeta}^k)$ follows an approach similar to that used for the lower sequence $(\check{\zeta}^k)$. The goal here is to establish that the two sequences, $(\hat{\zeta}^k)$ and $(\check{\zeta}^k)$, maintain an ordering relationship across all subdomains. Specifically, we aim to show:

$$\check{\zeta}^k \leq \hat{\zeta}^k, \quad \forall k \in \mathbb{N}.$$

For the initial iteration $k = 0$, we assume that $\check{\zeta}^0 \leq \hat{\zeta}^0$. In the first subdomain, since $\check{\zeta}^1$ is a lower solution, it satisfies the inequality:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}^1_1 - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}^0_2 - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^0_1) \right\} \leq 0. \tag{12}$$

On the other hand, since $\hat{\zeta}^1$ is an upper solution, it must satisfy:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\hat{\zeta}^1_1 - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\hat{\zeta}^0_2 - \mathscr{G}^{\mathfrak{i}}(\hat{\zeta}^0_1) \right\} \geq 0. \tag{13}$$

By comparing the inequalities in equations (12) and (13), we deduce that:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\check{\zeta}^1_1 - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\check{\zeta}^0_2 - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^0_1) \right\} \leq \max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ \mathscr{B}^{\mathfrak{i}}_{1,1}\hat{\zeta}^1_1 - \tilde{\mathscr{B}}^{\mathfrak{i}}_{1,2}\hat{\zeta}^0_2 - \mathscr{G}^{\mathfrak{i}}(\hat{\zeta}^0_1) \right\}.$$

Thus, we obtain $\check{\zeta}^0_2 \leq \hat{\zeta}^0_2$ and $\check{\zeta}^0_1 \leq \hat{\zeta}^0_1$, leading to:

$$\check{\zeta}^1_1 \leq \hat{\zeta}^1_1.$$

Next, solving the problem in the second subdomain gives the following inequalities:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}^1_1 + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}^1_2 - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^0_2) \right\} \leq 0, \tag{14}$$

and

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\hat{\zeta}^1_1 + \mathscr{B}^{\mathfrak{i}}_{2,2}\hat{\zeta}^1_2 - \mathscr{G}^{\mathfrak{i}}(\hat{\zeta}^0_2) \right\} \geq 0. \tag{15}$$

By comparing equations (14) and (15), we obtain:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\check{\zeta}^1_1 + \mathscr{B}^{\mathfrak{i}}_{2,2}\check{\zeta}^1_2 - \mathscr{G}^{\mathfrak{i}}(\check{\zeta}^0_2) \right\} \leq \max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}^{\mathfrak{i}}_{2,1}\hat{\zeta}^1_1 + \mathscr{B}^{\mathfrak{i}}_{2,2}\hat{\zeta}^1_2 - \mathscr{G}^{\mathfrak{i}}(\hat{\zeta}^0_2) \right\}.$$

From the assumption $\check{\zeta}_1^1 \leq \hat{\zeta}_1^1$ and $\check{\zeta}_2^0 \leq \hat{\zeta}_2^0$, we conclude that:

$$\check{\zeta}_2^1 \leq \hat{\zeta}_2^1.$$

Therefore, we can conclude that:

$$\check{\zeta}^1 \leq \hat{\zeta}^1.$$

By applying induction, we prove that $\check{\zeta}^k \leq \hat{\zeta}^k$ for all k.

Since the sequence $(\check{\zeta}^k)$ is increasing and bounded above, while the sequence $(\hat{\zeta}^k)$ is decreasing and bounded below, and given that these two sequences are interdependent, it can be concluded that both sequences converge to the unique solution of the discrete problem (3). Thus, we obtain:

$$\lim_{k\to\infty} \hat{\zeta}^k = \lim_{k\to\infty} \check{\zeta}^k = \zeta$$

Theorem 2. *For any initial solution ζ^0, the iterative sequence (ζ^k), generated by the algorithm converges geometrically to the unique solution of the system (3), with a positive constant $0 < \eta < 1$ such that:*

$$\|\zeta^{k+1} - \zeta^k\|_{L^\infty(\Omega)} \leq C\eta^k, \quad k \in \mathbb{N}.$$

Proof. We begin by examining the formulation of the system and noting the following relations:

$$\max_{1\leq i\leq n} \left\{ \mathscr{B}^{i}_{1,1}\zeta_1^{k+1} - \tilde{\mathscr{B}}^{i}_{1,2}\zeta_2^k - \mathscr{G}^{i}(\zeta_1^k) \right\} = 0,$$

and

$$\max_{1\leq i\leq n} \left\{ \mathscr{B}^{i}_{1,1}\zeta_1^{k} - \tilde{\mathscr{B}}^{i}_{1,2}\zeta_2^{k-1} - \mathscr{G}^{i}(\zeta_1^{k-1}) \right\} = 0.$$

These conditions correspond to solving the following problems:

$$\begin{cases} \max\limits_{1\leq i\leq n} \left(\mathscr{B}^{i}_{1,1}\zeta_1^{k+1} - \mathscr{G}^{i}(\zeta_1^k) \right) = 0, & \text{in } \Omega_1, \\ \left(\zeta_1^{k+1}\right)_\beta = \left(\zeta_2^k\right)_\beta, & \text{on } \Sigma_1, \end{cases}$$

and

$$\begin{cases} \max\limits_{1\leq i\leq n} \left(\mathscr{B}^{i}_{1,1}\zeta_1^{k} - \mathscr{G}^{i}(\zeta_1^{k-1}) \right) = 0, & \text{in } \Omega_1, \\ \left(\zeta_1^{k}\right)_\beta = \left(\zeta_2^{k-1}\right)_\beta, & \text{on } \Sigma_1. \end{cases}$$

By analyzing the boundary conditions of both problems and using the results from [10] applied to the Hamilton-Jacobi-Bellman equation, we deduce the existence of a constant $\kappa_1 \in (0, 1)$ such that:

$$\kappa_1 = \sup\{v(x) : x \in \Sigma_1\}.$$

Thus, we obtain the following inequality:

$$\|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Sigma_1)} \leq \kappa_1 \|\zeta_2^k - \zeta_2^{k-1}\|_{L^\infty(\Omega_2)}.$$

Similarly, analyzing the second part of the system, we have:

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}_{2,1}^{\mathfrak{i}} \zeta_1^{k+1} + \mathscr{B}_{2,2}^{\mathfrak{i}} \zeta_2^{k+1} - \mathscr{G}^{\mathfrak{i}}(\zeta_2^k) \right\} = 0,$$

and

$$\max_{1\leq \mathfrak{i}\leq \mathfrak{n}} \left\{ -\tilde{\mathscr{B}}_{2,1}^{\mathfrak{i}} \zeta_1^{k} + \mathscr{B}_{2,2}^{\mathfrak{i}} \zeta_2^{k} - \mathscr{G}^{\mathfrak{i}}(\zeta_2^{k-1}) \right\} = 0.$$

By applying the results from [10], we conclude the existence of a constant $\kappa_2 \in (0,1)$ such that:

$$\kappa_2 = \sup\{v(x) : x \in \Sigma_2\}.$$

This implies the following relationship:

$$\|\zeta_2^{k+1} - \zeta_2^k\|_{L^\infty(\Sigma_2)} \leq \kappa_2 \|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Omega_1)}.$$

By applying the maximum principle, we obtain:

$$\begin{aligned}
\|\zeta_2^{k+1} - \zeta_2^k\|_{L^\infty(\Omega_2)} &\leq \|\zeta_2^{k+1} - \zeta_2^k\|_{L^\infty(\Sigma_2)}, \\
\|\zeta_2^{k+1} - \zeta_2^k\|_{L^\infty(\Sigma_2)} &\leq \kappa_2 \|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Omega_1)}, \\
&\leq \kappa_2 \|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Sigma_1)}, \\
&\leq \kappa_1\kappa_2 \|\zeta_2^{k} - \zeta_2^{k-1}\|_{L^\infty(\Omega_2)}.
\end{aligned}$$

Similarly, we derive:

$$\begin{aligned}
\|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Omega_1)} &\leq \|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Sigma_1)}, \\
\|\zeta_1^{k+1} - \zeta_1^k\|_{L^\infty(\Sigma_1)} &\leq \kappa_1 \|\zeta_2^{k} - \zeta_2^{k-1}\|_{L^\infty(\Omega_2)}, \\
&\leq \kappa_1 \|\zeta_2^{k} - \zeta_2^{k-1}\|_{L^\infty(\Sigma_2)}, \\
&\leq \kappa_1\kappa_2 \|\zeta_1^{k} - \zeta_1^{k-1}\|_{L^\infty(\Omega_1)}.
\end{aligned}$$

Finally, we conclude with the inequality:

$$\|\zeta^{k+1} - \zeta^k\|_{L^\infty(\Omega)} \leq \kappa_1\kappa_2 \|\zeta^k - \zeta^{k-1}\|_{L^\infty(\Omega)}.$$

By induction and noting that $\eta = \kappa_1\kappa_2 \in (0,1)$, we conclude:

$$\|\zeta^{k+1} - \zeta^k\|_{L^\infty(\Omega)} \leq \eta^k \|\zeta^1 - \zeta^0\|_{L^\infty(\Omega)}.$$

Thus, we obtain:

$$\|\zeta^{k+1} - \zeta^k\|_{L^\infty(\Omega)} \leq C\eta^k, \quad k \in \mathbb{N}.$$

Therefore, the sequence (ζ^k) converges geometrically to the solution of the system (3).

4. Numerical Applications and Computational Results

In this section, we formulate the Hamilton-Jacobi-Bellman (HJB) equation using an overlapping domain decomposition method. The computational domain $\Omega = [0,1]\times[0,1]$ is discretized on a uniform grid with step size $\mathfrak{h}$ based on the finite difference method. The initial solution can be selected as either a lower or upper solution, both ensuring convergence to the discrete solution. The iterative process continues until the stopping criterion $\varepsilon = 10^{-6}$ is satisfied. The system is governed by the following equations:

$$\begin{cases} \max\left\{\mathscr{B}^1\zeta - \mathscr{G}^1, \mathscr{B}^2\zeta - \mathscr{G}^2\right\} \quad \text{in } \Omega, \\ \zeta = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{16}$$

To facilitate the numerical treatment, we introduce the following notations:

- $\mathscr{B}^1 = \mathscr{B} + \mu I$, $\mathscr{G}^1 = \mathscr{G} + \mu\zeta$, with $\mu = 1$.
- $\mathscr{B}^2 = I$, where I is the identity matrix, and $\mathscr{G}^2 = 0$.

The operator $\mathscr{B}$, which is non-coercive, is defined as:

$$\mathscr{B} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 0.5x\frac{\partial}{\partial x} + 0.5y\frac{\partial}{\partial y} + 0.045.$$

The function $\mathscr{G}$ is given by:

$$\mathscr{G} = \sin(2\pi x)\sin(2\pi y).$$

To evaluate the efficiency of the domain decomposition methods, we compare the performance of the non-overlapping domain decomposition ($\tau = 1$) (See [11]) with the newly introduced overlapping domain decomposition method ($\tau > 1$). This comparative study aims to assess the accuracy and convergence of both methods, verifying their effectiveness against theoretical expectations.
The numerical results, illustrating the efficiency of these approaches, are presented in tables and figures generated using MATLAB - R2018a.

Table 1: **Residual error using DDM for various overlap values.**

	$\|R\|_\infty$	
iteration	DDM: $\tau > 1$	DDM: $\tau = 1$
1	7.391048136608375e-04	9.591296770784198e-04
4	1.608036639652848e-04	2.363530936250824e-04
8	3.630680843518114e-05	5.836710081515498e-05
Last iteration	2.162254986382806e-05	3.666961301803263e-05

The table (1) presents the infinity norm of the residual error for the approximate solutions obtained using finite difference methods with $\mathfrak{h} = \frac{1}{20}$ under various overlap configurations. The residual is computed as

$$R = \max_{1\leq \mathrm{i}\leq 2}\left\{\mathscr{B}^{\mathrm{i}}\check{\zeta}^k - \mathscr{G}^{\mathrm{i}}\left(\check{\zeta}^{k-1}\right)\right\}.$$

The results indicate that the residual errors are remarkably small and consistently decrease across iterations. Notably, configurations with overlap ($\tau > 1$) result in a faster reduction in error compared to the non-overlapping case ($\tau = 1$). This emphasizes the importance of overlap in accelerating convergence and improving the accuracy of the solution approximation.

Table 2: **Value of** $\hat{\zeta}^k$ **at** $(x, y)^T = (0.5, 0.25)^T$ **for** $\mathfrak{h} = \frac{1}{20}$**.**

Iteration	DDM : $\tau > 1$	DDM : $\tau = 1$
1	-0.003915453822399	-0.003375478664288
4	-0.005478666922059	-0.005035539613455
8	-0.005493072486538	-0.005428621964170
Last iteration	-0.005493166542087	-0.005493095359904

Table 3: **Value of** $\check{\zeta}^k$ **at** $(x, y)^T = (0.5, 0.25)^T$ **for** $\mathfrak{h} = \frac{1}{20}$**.**

Iteration	DDM : $\tau > 1$	DDM : $\tau = 1$
1	-0.137907986376806	-1.035248717772950
4	-0.012726797737372	-0.386671348701684
8	-0.005505924360092	-0.098756491226852
Last iteration	-0.005493173019551	-0.005493282229003

The numerical results presented in Tables (2) and (3) illustrate the convergence properties of the lower and upper solutions for the Hamilton-Jacobi-Bellman (HJB) equation. As expected, the lower solution $\check{\zeta}^k$ exhibits a monotonic increase, while the upper solution $\hat{\zeta}^k$ follows a monotonic decrease, ensuring a stable convergence towards the exact solution. Moreover, the comparison between the overlapping ($\tau > 1$) and non-overlapping ($\tau = 1$) domain decomposition methods confirms that increasing the overlap parameter significantly enhances the convergence rate, reducing the number of iterations required to reach the desired accuracy.

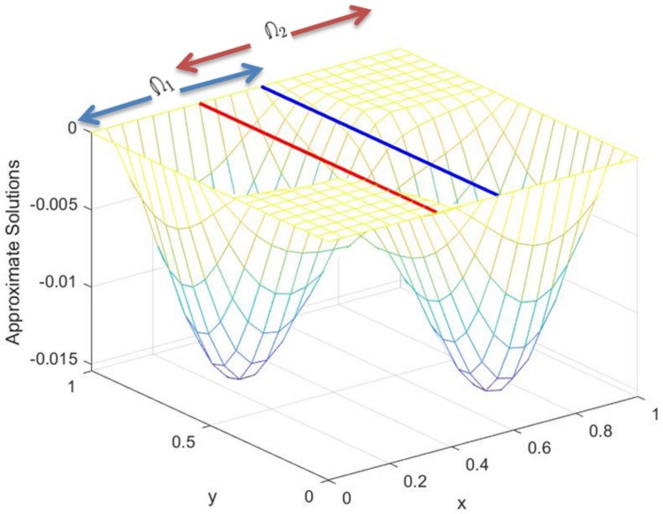

Figure 2: **Approximated Solutions (DDM).**

Figure (2) represents the numerical solution of the problem using the overlapping Schwarz domain decomposition method. The red and blue lines indicate the overlapping subdomains along the x-axis.

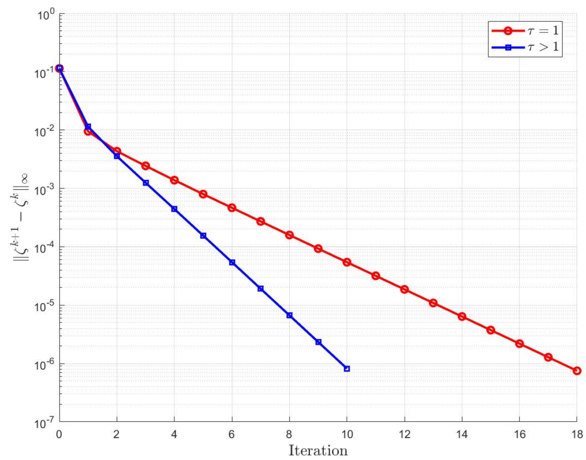

Figure 3: **Convergence Behavior of the Iterative Scheme.**

Figure (3) demonstrates the geometric convergence of the iterative method for $\mathfrak{h} = \frac{1}{20}$. The graph displays the evolution of the infinity norm of the error $\|\zeta^{k+1} - \zeta^k\|_\infty$ as a function of the iteration number. We observe a rapid exponential decrease in the error, indicating that the approximate solution is approaching a stable fixed point. Furthermore, using a larger overlap (blue curve) significantly improves the convergence rate by reducing both the number of iterations and the total computational effort.

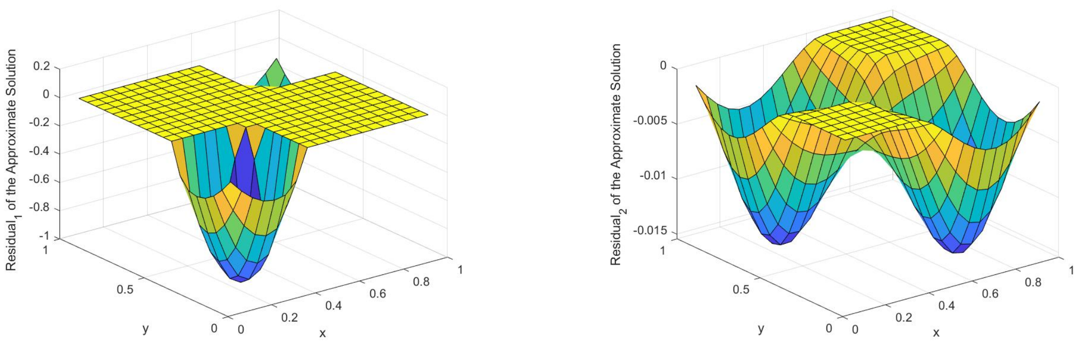

Figure 4: **Residuals Evolution for the Overlapping Domain Decomposition .**

Figure (4) illustrates the residuals R_1 and R_2, representing the error distribution in the approximate solution while highlighting the active and inactive points at the boundaries. These plots demonstrate the error evolution through iterations within each subdomain, where these values are utilized to adjust and refine the solution at each step.

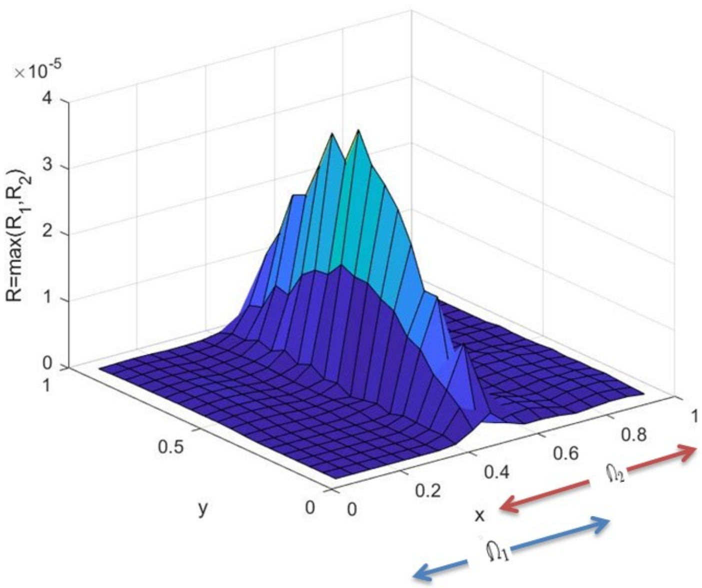

Figure 5: **Maximum Residual Over the Computational Domain.**

Figure (5) illustrates the distribution of the maximum residual over the computational domain. The error is clearly concentrated in the overlapping region between the two subdomains Ω_1 and Ω_2, where it drops to a minimal value of approximately 10^{-5}. This confirms the efficiency of the iterative method in reducing the residual specifically within the interface area, thereby enhancing the accuracy of the solution and accelerating convergence.

5. Conclusion

In this work, we solved the noncoercive Hamilton-Jacobi-Bellman equation directly, without transforming it into a quasi-variational inequality system. This approach preserves the structure of the original problem and simplifies the numerical treatment. The method is based on constructing lower and upper solutions that converge to the discrete solution within the finite difference framework.

The overlapping domain decomposition method was applied, which contributed to improving the convergence rate compared to the non-overlapping case. Theoretical analysis confirms that this method ensures monotonic and geometric convergence. Furthermore, numerical results demonstrated that the performance achieved by this method is characterized by high accuracy and exhibits significant similarity to methods such as the Two-Level and Multi-Grid approaches, in terms of convergence speed and result precision.

In future work, we plan to extend this method to q overlapping subdomains in higher-dimensional spaces, particularly for non-matching grids. Additionally, we plan to combine finite element discretizations with different boundary conditions, integrating the domain decomposition method with nonlinear multigrid techniques to further improve efficiency and scalability for large-scale problems. We will also investigate parallel implementations to enhance performance and tackle more complex problems.

Acknowledgements

The authors thank the reviewers and editors for their valuable feedback and support.

References

[1] L'ubomír Baňas, Herbert Dawid, Tsiry Avisoa Randrianasolo, Johannes Storn, and Xingang Wen. Numerical approximation of a system of Hamilton–Jacobi–Bellman equations arising in innovation dynamics. *Journal of Scientific Computing*, 92:54, 2022.

[2] Dietmar Gallistl and Endre Süli. Mixed Finite Element Approximation Of The Hamilton–Jacobi–Bellman Equation With Cordes Coefficients. *SIAM Journal on Numerical Analysis*, 57(2):592–614, 2019.

[3] Mohammed Beggas and Mohamed Haiour. The Maximum Norm Analysis of Schwarz Method for Elliptic Quasi-Variational Inequalities. *Kragujevac Journal of Mathematics*, 45(4):635–645, 2021.

[4] L. Djemaoune and M. Haiour. Overlapping Domain Decomposition Method for a Noncoercive System of Quasi-Variational Inequalities Related to the Hamilton–Jacobi–Bellman Equation. *Computational Mathematics and Modeling*, 27(2), 2016.

[5] Madjda Miloudi, Samira Saadi, and Mohamed Haiour. Hamilton-Jacobi-Bellman Equations: An Algorithmic Contraction New Approach. *Applied Mathematics & Information Sciences*, 15(1):83–88, 2021.

[6] Philippe Cortey-Dumont. Sur les inéquations variationnelles à opérateur non coercif. *Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 19(2):195–212, 1985.

[7] P. G. Ciarlet and P. A. Raviart. Maximum principle and uniform convergence for the finite element method. *Computer Methods in Applied Mechanics and Engineering*, 2(1):17–31, 1973.

[8] P. L. Lions and J. L. Menaldi. Optimal Control of Stochastic Integrals and Hamilton-Jacobi-Bellman Equations (Part I). *SIAM Journal on Control and Optimization*, 20:58–81, 1982.

[9] M. Boulbrachene and M. Haiour. The finite element approximation of Hamilton-Jacobi-Bellman equations. *Computers and Mathematics with Applications*, 41(7-8):993–1007, 2001.

[10] Mohamed Haiour and Salah Boulaaras. Overlapping domain decomposition methods for elliptic quasi-variational inequalities related to impulse control problem with mixed boundary conditions. *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)*, 121(4):481–493, 2011.

[11] M. Sun. Domain decomposition algorithms for solving hamilton-jacobi-bellman equations. *Numerical Functional Analysis and Optimization*, 14(1-2), 1993.

[12] M. Boulbrachene. A contraction approach for noncoercive hamilton-jacobi-bellman equations. *Applied Mathematics Information Sciences*, 4(2):261–269, 2010.

[13] M. Boulbrachene, M. Haiour, and B. Chentouf. On a Noncoercive System of Quasi-Variational Inequalities Related to Stochastic Control Problems. *Journal of Inequalities in Pure and Applied Mathematics*, 3(2):Article 30, 14 p, 2002.

[14] Salah Boulaaras and Mohamed Haiour. A General Case for the Maximum Norm Analysis of an Overlapping Schwarz Methods of Evolutionary HJB Equation with Nonlinear Source Terms with the Mixed Boundary Conditions. *Applied Mathematics & Information Sciences*, 9(3):1247–1257, 2015.

[15] Mohamed Haiour and Salah Boulaaras. Uniform Convergence of Schwarz Method for Elliptic Quasi-Variational Inequalities Related to Impulse Control Problem. *An-Najah University Journal of Research (N. Sc.)*, 24, 2010.

[16] Salah Boulaaras, Mohamed El Amine Bencheikh Le Hocine, and Mohamed Haiour. A new error estimate on uniform norm of a parabolic variational inequality with nonlinear source terms via the subsolution concepts. *Journal of Inequalities and Applications*, 2020:78, 2020.

[17] Ph. Cortey-Dumont. Sur l'analyse Numérique des Équations de Hamilton-Jacobi-Bellman. *Mathematical Methods in the Applied Sciences*, 9:198–209, 1987.

[18] Lawrence C. Evans and Avner Friedman. Optimal Stochastic Switching And The Dirichlet Problem For The Bellman Equations. *Transactions of the American Mathematical Society*, 253:365–389, 1979.

[19] M. J. Gander and L. Halpren. Méthodes de décomposition de domaines - notions

de base. *Sciences Fondamentales Mathematics*, AF1375(V1), 2012.

[20] Abida Harbi, Nasreddine Nemis, and Mohamed Haiour. Optimal error estimates of a class of system of two quasi-variational inequalities. *AIMS Mathematics*, 6(6):5977–6002, 2021.

[21] Ronald H. W. Hoppe. Multi-grid methods for Hamilton-Jacobi-Bellman equations. *Numerische Mathematik*, 49:239–254, 1986.

[22] P. L. Lions and B. Mercier. Approximation numérique des Équations de Hamilton-Jacobi-Bellman. *RAIRO. Analyse Numérique*, 14(4):369–393, 1980.

[23] Madjda Miloudi, Samira Saadi, and Mohamed Haiour. L^{∞}-error estimates of a finite element method for Hamilton–Jacobi–Bellman equations with nonlinear source terms with mixed boundary condition. *Demonstratio Mathematica*, 54(1):452–461, 2021.

[24] P. Spiteri, J. C. Miellou, and D. El Baz. Asynchronous Schwarz alternating methods with flexible communication for the obstacle problem. *Calculateurs Parallèles, Réseaux et Systèmes Répartis*, 13(1):47–66, 2001.