EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6072
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Common Riccati Stability and Time-Delay Systems

Ali Algefary[1,*], Khulud Abdullah Alqufari[2]

[1] *Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia.*
[2] *Department of Statistics and Operation Research, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia*

Abstract. This paper investigates the stability properties of matrix families through the lens of Riccati, Lyapunov, and Schur stability. We focus on establishing connections between these stability concepts, particularly in the context of continuous and discrete-time systems, as well as time-delay systems. The results provide criteria for common Riccati stability, exploring its implications on Lyapunov and Schur stability across matrix families. Furthermore, we examine the effects of scaling transformations and similarity transformations on common Riccati stability, demonstrating its robustness under scalar multiplication and similarity changes. The findings contribute to a deeper understanding of matrix stability in control systems, offering insights into the structural preservation of stability properties under various transformations. This work has potential applications in robust control and the stability analysis of complex systems subject to time delays and structural modifications.

2020 Mathematics Subject Classifications: 15A45, 15B48, 93D05, 34K25, 34K05

Key Words and Phrases: Common Riccati Stability, differential equations, Lyapunov functions, positive definite matrices, Common Lyapunov stability

1. Introduction

The work in this paper considers only matrices in the real space $\mathbb{R}^{n\times n}$. We use the notation $X \succ 0$ ($X \prec 0$, respectively) to indicate that a matrix $X \in \mathbb{R}^{n\times n}$ is positive definite (negative definite, respectively). Similarly, we denote a positive semidefinite (negative semidefinite, respectively) matrix by $X \succeq 0$ ($X \preceq 0$, respectively). Unless stated otherwise, a positive or negative definite or semidefinite matrix is assumed to be symmetric.

For a matrix $X \in \mathbb{R}^{n\times n}$, we write X^T to denote the transpose of X, we also write X^{-1} to refer to the inverse of X, and X^{-T} for its inverse transpose, defined as $X^{-T} = (X^T)^{-1} = (X^{-1})^T$. Additionally, we use I to denote the identity matrix with its size implied by the context.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6072

Email addresses: a.algefary@qu.edu.sa (A. Algefary), 432206823@qu.edu.sa (K. A. Alqufari)

Consider a matrix $X \in \mathbb{R}^{n\times n}$. In this paper, we use $\sigma(X)$ to represent the spectrum of X and $\rho(X)$ to indicate its spectral radius. The spectral abscissa of X, represented by $\alpha(X)$, is defined as the largest real part among the eigenvalues of X. Formally, we write:

$$\alpha(X) = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(X)\}.$$

For further details, refer to [1, 2].

Let us start by reviewing some key definitions related to types of matrix stability relevant to this study.

Definition 1. *[3] Let A be a matrix in $\mathbb{R}^{n\times n}$. The matrix A is called Hurwitz (or Hurwitz stable) if all its eigenvalues are located in the open left half of the complex plane, meaning $\alpha(A) < 0$.*

A well-known characterization of Hurwitz stability for a matrix $A \in \mathbb{R}^{n\times n}$ involves the Lyapunov equation

$$A^T P + PA + Q = 0,$$

where $P, Q \in \mathbb{R}^{n\times n}$ with $P \succ 0$ and $Q \succ 0$; see [3]. This result is fundamental in assessing the stability of continuous linear systems described by

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $x(t) \in \mathbb{R}^n$. Specifically, if there exists a $P \succ 0$ satisfying the Lyapunov equation, then the linear system in (1) is associated with a Lyapunov function $V(x) = x^T Px$, confirming the system equilibrium is asymptotic stability. This criterion for Hurwitz stability is presented in the following lemma, known as Lyapunov's Theorem.

Lemma 1. [3, 4] *A matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz stable if there exist positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ such that*

$$A^T P + PA = -Q.$$

Alternatively, Lemma 1 can be restated using the Lyapunov inequality: A is Hurwitz stable if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A^T P + PA \prec 0. \tag{2}$$

Within this context, P is referred to as a Lyapunov solution for matrix A or the Lyapunov inequality.

The concept of Hurwitz stability for a single matrix $A \in \mathbb{R}^{n\times n}$ has been extended in the literature (see [5–8]) to consider families of real matrices through the notion of common Lyapunov stability. This extension is particularly significant in control theory, especially in the analysis of switched systems and robustness, where ensuring stability across multiple system configurations is crucial [9, 10].

Specifically, a family of real matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$ is said to have common Lyapunov stability if there exists a positive definite matrix P such that for each $i = 1, \ldots, m$,

$$A_i^T P + P A_i \prec 0. \tag{3}$$

The existence of such a matrix P implies simultaneous Hurwitz stability for the matrices in $\mathcal{A}$. In other words, the linear systems associated with the matrices in the family $\mathcal{A}$ share a common Lyapunov function given by

$$V(x) = x^T P x, \tag{4}$$

which serves as a unified measure to demonstrate the stability of each system in the family.

The concepts of Hurwitz stability and common Lyapunov stability have been further extended to diagonal Lyapunov stability [4, 11, 12] and common diagonal Lyapunov stability [4, 13, 14], respectively. This extension focuses on identifying a diagonal positive definite matrix $P = \operatorname{diag}(p_1, p_2, \ldots, p_n)$ such that, for a single matrix A, the Lyapunov inequality (2) is satisfied. This formulation simplifies the task by narrowing it down to finding the positive diagonal elements p_i, taking advantage of the diagonal structure for easier computations.

In a similar way, for a family of matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$, common diagonal Lyapunov stability entails finding a single diagonal positive definite matrix P that meets the inequalities (3). In this case, P is referred to as a common Lyapunov solution for the family $\mathcal{A}$. Equivalently, the family $\mathcal{A}$ has common Lyapunov stability if there exist positive definite matrices $P, Q_1, \ldots, Q_m \in \mathbb{R}^{n \times n}$ such that

$$A_i^T P + P A_i + Q_i = 0,$$

for $i = 1, \ldots, m$.

This approach proves especially valuable in large-scale systems or in cases involving decoupled or weakly coupled subsystems, where computational simplicity and efficiency are key. By restricting P to be diagonal, we lower the problem's complexity while maintaining a unified framework for proving the stability of each system in the family.

Building upon the continuous-time case, we now consider discrete-time difference systems, which are fundamental in digital control and signal processing [15]. Specifically, we examine the system:

$$x(k+1) = Ax(k).$$

The zero equilibrium of this system is asymptotically stable if there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A^T P A - P \prec 0.$$

This condition ensures that the Lyapunov function $V(x) = x^T Px$ decreases over time, guaranteeing stability. Similar to the continuous-time scenario, the stability of the discrete-time system can also be characterized by the eigenvalues of A. Specifically, the system is asymptotically stable if all eigenvalues of A lie strictly inside the open unit disk of the complex plane. When this condition is met, A is referred to as Schur stable matrix.

Definition 2. *[4] A matrix $B \in \mathbb{R}^{n\times n}$ is called Schur stable if its spectral radius satisfies $\rho(B) < 1$.*

Lemma 2. *[4] A matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable if and only if there exist positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ such that:*

$$A^T PA - P = -Q.$$

This equation is known as the Stein equation and plays a significant role in discrete-time stability analysis, analogous to the Lyapunov equation in continuous-time systems.

Extending this notion to the common case, consider a family of matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. The family $\mathcal{A}$ is said to possess common Schur stability if there exists a single positive definite matrix P such that for all $i = 1, \ldots, m$, the following inequality holds

$$A_i^T PA_i - P \prec 0.$$

This P is known as the common Schur solution for the family $\mathcal{A}$. The definition of common Schur stability can stated using an equation instead of inequality. That is $\mathcal{A}$ has a common Schur stability if there exist positive definite matrices $P, Q_1, \ldots, Q_m \in \mathbb{R}^{n\times n}$ such that

$$A_i^T PA_i - P = -Q_i,$$

$i = 1, \ldots, m$.

The existence of such a common matrix P implies that each system in the family, described by $x(k + 1) = A_i x(k)$, is asymptotically stable. Moreover, they all share the common Lyapunov function as in (4) which serves as a unified tool to demonstrate the stability of all systems within the family $\mathcal{A}$.

Moving forward, consider the linear differential systems with time delays, which are frequent in many engineering applications where delays are inevitable. Specifically, we examine the system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \tag{5}$$

where $A, B \in \mathbb{R}^{n\times n}$ are constant matrices, $x(t) \in \mathbb{R}^n$ is the state vector, and $\tau \geq 0$ represents an arbitrary time delay. Understanding the stability of such time-delay systems is crucial because delays can significantly affect system performance and may lead to instability if not properly accounted for [16].

As demonstrated in [17], the system (5) admits a Lyapunov-Krasovskii functional of the form

$$V(x) = x^T P x + \int_{t-\tau}^{t} x^T(s) Q x(s)\, ds,$$

provided that there exist positive definite matrices $P, Q, R \in \mathbb{R}^{n\times n}$ satisfying the Riccati equation

$$A^T P + PA + Q + PBQ^{-1}B^T P + R = 0.$$

The existence of this Lyapunov-Krasovskii functional is significant because it guarantees that the equilibrium point of system (5) is asymptotically stable for all delays $\tau \geq 0$ see [18].

The requirement that positive definite matrices P, Q, R satisfy the Riccati equation defines the concept of Riccati stability for a pair of matrices (A, B), which was introduced in [19]. The connections between Riccati stability and classical notions of stability, such as Hurwitz and Schur stability, have been explored in [20]. Moreover, several results establish links between Riccati stability and the stability analysis of time-delay systems, underscoring its importance in this area of study.

Similar to the concept of common Lyapunov stability, we introduce the notion of *common Riccati stability*. This concept involves finding positive definite matrices $P, Q, R_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, that satisfy a Riccati equation simultaneously for a family of matrix pairs $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$.

Definition 3. *Let $A_i, B_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, m$. We say that the family of pairs $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ has common Riccati stability if there exist positive definite matrices $P, Q, R_i \in \mathbb{R}^{n\times n}$ such that the following Riccati equations*

$$A_i^T P + PA_i + Q + PB_iQ^{-1}B_i^T P + R_i = 0,$$

$i = 1, \ldots, m$, hold.

When such matrices P, Q, R_i, $i = 1, \ldots, m$, exist, we refer to the pair (P, Q) as a common Riccati solution for the family $\mathcal{U}$. Essentially, common Riccati stability represents a simultaneous solution to the Riccati equations associated with each pair (A_i, B_i) in $\mathcal{U}$.

The existence of a common Riccati solution for the family $\mathcal{U}$ implies that

$$V(x) = x^T P x + \int_{t-\tau}^{t} x^T(s) Q x(s)\, ds$$

serves as a common Lyapunov-Krasovskii functional for all time-delay systems associated with the pairs in $\mathcal{U}$.

The main contributions of this paper are organized into two key parts. In the first part, we explore the connections between common Riccati stability for a family of matrix pairs $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ and the concepts of common Lyapunov stability and common Schur stability. By investigating these relationships, we aim to deepen the understanding of how these different stability notions are interrelated within the context of control theory and system dynamics.

In the second part, we explore several scaling properties associated with common Riccati stability. Analyzing these properties is significant because scaling can affect the stability of systems, and understanding this impact is crucial for the design and analysis of robust control systems. By identifying how scaling transformations influence common Riccati stability, we provide insights that can lead to more efficient computational methods and enhance the applicability of stability criteria to a broader class of systems.

2. Common Lyapunov, Schur, and Riccati stability

In this section, we explore the foundational relationships among common Lyapunov stability, common Schur stability, and common Riccati stability for families of matrices and matrix pairs. These stability concepts play crucial roles in analyzing and ensuring the robustness of complex systems, particularly in control theory. By developing these interconnections, we provide a unified framework that simplifies stability analysis across various system structures and enhances our understanding of stability in both continuous and discrete-time systems.

Theorem 1. *For $i = 1, \ldots, m$, suppose that $A_i, B_i \in \mathbb{R}^{n\times n}$. If the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ has a common Riccati stability, then the family $\mathcal{A} = \{A_i\}_{i=1}^m$ has a common Lyapunov stability.*

Proof. Let $P, Q, R_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$ be positive definite matrices such that (P, Q) is a common Riccati solution for the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$. Then, for each $i = 1, \ldots, m$, the matrices

$$X_i = Q + PB_iQ^{-1}B_i^TP + R_i$$

are positive definite. Consequently, we have

$$A_i^TP + PA_i + X_i = 0$$

for every $i \in \{1, \ldots, m\}$. Thus, P serves as a common Lyapunov solution for the family $\mathcal{A} = \{A_i\}_{i=1}^m$.

Theorem 2. *For $i = 1, \ldots, m$, suppose that $A_i, B_i \in \mathbb{R}^{n\times n}$. If the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ has common Riccati stability, then the family $\{A_i^{-1}B_i\}_{i=1}^m$ has common Schur stability.*

Proof. Let $P, Q, R_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$ be positive definite matrices such that (P, Q) is a common Riccati solution for the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$. Then, for $i = 1, \ldots, m$, the following equations hold

$$A_i^TP + PA_i + Q + PB_iQ^{-1}B_i^TP + R_i = 0. \tag{6}$$

According to Theorem 1, the family $\{A_i\}_{i=1}^m$ has common Lyapunov stability, and therefore, each A_i is nonsingular. Thus, the equations in (6) can be rewritten as

$$A_i^TP + PA_i + Q + PA_i\left[A_i^{-1}B_iQ^{-1}B_i^TA_i^{-T}\right]A_i^TP + R_i = 0$$

for each i. Next, observe that for each $i \in \{1, \dots, m\}$, these equations are identical to

$$(PA_iQ^{-1} + I)Q(I + Q^{-1}A_i^TP) + PA_i\left[A_i^{-1}B_iQ^{-1}B_i^TA_i^{-T} - Q^{-1}\right]A_i^TP + R_i = 0. \quad (7)$$

Define

$$Z_i = (PA_iQ^{-1} + I)Q(I + Q^{-1}A_i^TP) + R_i,$$

for $i \in \{1, \dots, m\}$. It is evident that each of Z_i is a positive definite matrix. By substituting Z_i in (7), we obtain

$$PA_i\left[A_i^{-1}B_iQ^{-1}B_i^TA_i^{-T} - Q^{-1}\right]A_i^TP + Z_i = 0.$$

For $i = 1, \dots, m$, note that A_i is nonsingular, and since P is positive definite, it is also nonsingular. Therefore, the matrices PA_i and $(PA_i)^T$ for $i = 1, \dots, m$ are nonsingular. Now, observe that for $i = 1, \dots, m$, we have $(PA_i)^{-1} = A_i^{-1}P^{-1}$ and $(PA_i)^{-T} = P^{-1}A_i^{-T}$. Consequently, for each $i \in \{1, \dots, m\}$, we can pre-multiply the i^{th} equation in (7) by $(PA_i)^{-1}$ and post-multiply it by $(PA_i)^{-T}$. Thus, the equations in (7) become

$$A_i^{-1}B_iQ^{-1}B_i^TA_i^{-T} - Q^{-1} + (PA_i)^{-1}Z_i(PA_i)^{-T} = 0, \quad (8)$$

for $i = 1, \dots, m$. Next, let

$$Y_i = (PA_i)^{-1}Z_i(PA_i)^{-T},$$

for $i = 1, \dots, m$. Recall that each Z_i is positive definite; consequently, Y_i is positive definite for each $i \in \{1, \dots, m\}$. Hence, the equations in (8) reduce to the following

$$(A_i^{-1}B_i)Q^{-1}(A_i^{-1}B_i)^T - Q^{-1} + Y_i = 0,$$

for $i = 1, \dots, m$. By the definition of common Schur stability, this implies that the family $\{A_i^{-1}B_i\}_{i=1}^m$ has common Schur stability.

Having established that common Riccati stability implies both common Lyapunov and Schur stability, we now examine the reverse relationship in Theorems 3 and 4. Specifically, in Theorem 3, we show that if the family $\left\{A_i^{-1}B_i\right\}_{i=1}^m$ possesses common Schur stability, then there exist orthogonal transformations of the matrices A_i that ensure each transformed matrix is Hurwitz. Furthermore, under these transformations, the pairs $(\Delta_iA_i, \Delta_iB_i)$ maintain a Riccati solution that shares a common matrix.

Theorem 3. *For $i = 1, \dots, m$, suppose that $A_i, B_i \in \mathbb{R}^{n\times n}$. If the family $\{A_i^{-1}B_i\}_{i=1}^m$ has common Schur stability, then there exist orthogonal matrices Δ_i, $i = 1, \dots, m$, such that for each i, Δ_iA_i is Hurwitz, and for each i, the pair $(\Delta_iA_i, \Delta_iB_i)$ has a Riccati solution (X_i, Y), i.e. all the pairs share the same Y in their Riccati solution.*

Proof. As $\{A_i^{-1}B_i\}_{i=1}^m$ has common Schur stability, there are positive definite matrices $P, Q_1, \dots, Q_m$ such that

$$(A_i^{-1}B_i)P(A_i^{-1}B_i)^T + Q_i = P, \quad (9)$$

for $i = 1, \ldots, m$. Additionally, the existence of the family $\{A_i^{-1}B_i\}_{i=1}^m$ means that for each i, the matrices A_i^{-1} exist. Now, let $Y = P^{-1} \succ 0$ and suppose that the matrices YA_i^{-1} for all $i \in \{1, \ldots, m\}$ have singular value decomposition

$$YA_i^{-1} = U_i \Sigma_i V_i^T,$$

where $U_i U_i^T = I$, $V_i V_i^T = I$, and the diagonal matrices Σ_i are positive definite. Next, for each i, define $\Delta_i = -U_i V_i^T$. Thus, we have

$$YA_i^{-1}\Delta_i^T = -U_i \Sigma_i U_i^T = -X_i. \tag{10}$$

Clearly, each X_i is a positive definite matrix. Now, for each i, observe that

$$Y = -X_i \Delta_i A_i = -A_i^T \Delta_i^T X_i,$$

i.e.,

$$Y + X_i \Delta_i A_i = 0$$

and

$$Y + A_i^T \Delta_i^T X_i = 0.$$

Adding these two equations together, we obtain

$$(\Delta_i A_i)^T X_i + X_i(\Delta_i A_i) + 2Y = 0, \tag{11}$$

for $i = 1, \ldots, m$. This implies that for every $i \in \{1, \ldots, m\}$, $\Delta_i A_i$ is Hurwitz. Now, from (9), it follows that

$$A_i^{-1}\Delta_i^T \Delta_i B_i Y^{-1} B_i^T \Delta_i^T \Delta_i A_i^{-T} + Q_i = Y^{-1}.$$

By pre-multiplying this last equation by Y and post-multiplying it by Y^T, we get

$$YA_i^{-1}\Delta_i^T(\Delta_i B_i)Y^{-1}(B_i^T \Delta_i^T)\Delta_i A_i^{-T}Y^T + YQ_iY^T = Y,$$

$i = 1, \ldots, m$. This, by (10), is equivalent to

$$X_i(\Delta_i B_i)Y^{-1}(\Delta_i B_i)^T X_i + YQ_iY^T = Y, \tag{12}$$

$i = 1, \ldots, m$. Combining (11) and (12) gives

$$(\Delta_i A_i)^T X_i + X_i(\Delta_i A_i) + Y + X_i(\Delta_i B_i)Y^{-1}(\Delta_i B_i)^T X_i + Z_i = 0,$$

$i = 1, \ldots, m$, where $Z_i = YQ_iY^T$. Thus, the conclusion of the theorem holds.

Next, we extend the results on common Lyapunov solutions to demonstrate that, given a family of matrices with a common Lyapunov solution, we can construct a corresponding family of matrix pairs $\{(A_i, B_i)\}_{i=1}^m$ that shares a common Riccati solution. This result bridges the gap between Lyapunov stability for a family of matrices and Riccati stability for pairs of matrices, highlighting how solutions in one framework can lead to stability in the other. By constructing appropriate matrices B_i, we show that a single Lyapunov matrix can satisfy the Riccati equations across the family, establishing a unified stability criterion.

Theorem 4. *For $i = 1, \ldots, m$, suppose that $A_i \in \mathbb{R}^{n\times n}$. If the family $\mathcal{A} = \{A_i\}_{i=1}^m$ has a common Lyapunov solution, then there exist matrices B_i, $i = 1, \ldots, m$, such that for each i, the pair (A_i, B_i) has a Riccati solution $(P, \hat{Q}_i)$, i.e. all the pairs share the same P in their Riccati solution.*

Proof. Let us assume that the family $\mathcal{A}$ has a common Lyapunov solution. Then, there exist positive definite matrices $P, Q_1, \ldots, Q_m$ such that

$$A_i^T P + PA_i + Q_i = 0, \quad i = 1, \ldots, m.$$

Now, consider arbitrary scalars $\alpha > 0$ and $\beta > 0$ with $\alpha^2 + \beta^2 < 1$. Therefore, it follows that

$$A_i^T P + PA_i + \alpha^2 Q_i + (1-\alpha^2-\beta^2)Q_i + P\left(P^{-1}\alpha\beta Q_i\right)Q_i^{-1}\alpha^{-2}\left(Q_i\alpha\beta P^{-1}\right)P = 0, \quad i = 1, \ldots, m. \tag{13}$$

Define

$$B_i = \alpha\beta P^{-1} Q_i,$$

and

$$S_i = (1 - \alpha^2 - \beta^2)Q_i,$$

$i = 1, \ldots, m$. Then, equation (13) reduces to

$$A_i^T P + PA_i + \alpha^2 Q_i + S_i + PB_i Q_i^{-1}\alpha^{-2} B_i^T P = 0, \quad i = 1, \ldots, m.$$

On letting $\hat{Q}_i = \alpha^2 Q_i$, this implies that for each $i \in \{1, \ldots, m\}$, the pair (A_i, B_i) has $(P, \hat{Q}_i)$ as a Riccati solution. This completes the proof.

This section establishes the foundational relationships between common Riccati, Lyapunov, and Schur stability. Theorems 2.1 and 2.2 demonstrate that common Riccati stability implies both common Lyapunov and Schur stability, highlighting how the Riccati framework provides a unifying structure for ensuring system robustness across continuous and discrete-time settings. These results indicate that solving the Riccati equations for a family of matrix pairs yields solutions that inherently satisfy Lyapunov-type inequalities, thus extending classical notions of stability to more complex system structures. In the next section, we build upon these results by exploring the preservation of Riccati stability under various transformations, further emphasizing the robustness of the proposed approach.

3. Stability preservation in scaled common Riccati families

In this section, we explore how common Riccati stability behaves under transformations, focusing on scaling properties and similarity transformations. These adaptation examine how stability characteristics are preserved or modified when the family of matrix pairs undergoes specific alterations, such as scaling by a positive scalar or transformations by a nonsingular matrix. This analysis is crucial in control theory applications where systems may be subject to rescaling or coordinate transformations, yet stability needs to be maintained across these changes.

Theorem 5. *For $i = 1, \dots, m$, suppose that $A_i, B_i \in \mathbb{R}^{n\times n}$. If the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ has common Riccati stability, then the family $\mathcal{V} = \{(\beta A_i, \beta B_i)\}_{i=1}^m$ has common Riccati stability for all $\beta > 0$.*

Proof. Assume that there are positive definite matrices $P, Q, R_1, \dots, R_m \in \mathbb{R}^{n\times n}$ such that (P, Q) is a common Riccati solution for $\mathcal{U}$. Thus, the following equations hold

$$A_i^T P + PA_i + Q + PB_iQ^{-1}B_i^T P + R_i = 0,$$

for $i = 1, \dots, m$. Thus, for every $\beta > 0$, we have

$$\beta(A_i^T P + PA_i + Q + PB_iQ^{-1}B_i^T P + R_i) = 0,$$

for $i = 1, \dots, m$, which are equivalent to

$$(\beta A_i)^T P + P(\beta A_i) + \beta Q + P(\beta B_i)\frac{Q^{-1}}{\beta}(\beta B_i)^T P + (\beta R_i) = 0,$$

for $i = 1, \dots, m$. This implies that the pair $(P, \beta Q)$ is a common Riccati solution for the family $\mathcal{V}$.

Theorem 6. *For $i = 1, \dots, m$, suppose that $A_i, B_i \in \mathbb{R}^{n\times n}$. If the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ has common Riccati stability, then for any nonsingular matrix $H \in \mathbb{R}^{n\times n}$, the pair $(H^{-T}PH^{-1}, H^{-T}QH^{-1})$ is a common Riccati solution for the family $\mathcal{V} = \{(HA_iH^{-1}, HB_iH^{-1})\}_{i=1}^m$.*

Proof. Suppose that (P, Q) is a common Riccati solution for the $\mathcal{U}$. Thus, there are positive definite matrices $R_1, \dots, R_m \in \mathbb{R}^{n\times n}$ satisfying

$$A_i^T P + PA_i + Q + PB_iQ^{-1}B_i^T P + R_i = 0,$$

for $i = 1, \dots, m$. Consequently,

$$H^{-T}(A_i^T P + PA_i + Q + PB_iQ^{-1}B_i^T P + R_i)H^{-1} = 0,$$

for all i, and therefore, we have

$$H^{-T}A_i^T PH^{-1} + H^{-T}PA_iH^{-1} + H^{-T}QH^{-1} + H^{-T}PB_iQ^{-1}B_i^T PH^{-1} + H^{-T}R_iH^{-1} = 0.$$

Therefore, we get that

$$(HA_iH^{-1})^TH^{-T}PH^{-1} + H^{-T}PH^{-1}(HA_iH^{-1}) + H^{-T}QH^{-1} + \\ H^{-T}PH^{-1}(HB_iH^{-1})(H^{-T}QH^{-1})^{-1}(HB_iH^{-1})^TH^{-T}PH^{-1} + H^{-T}R_iH^{-1} = 0,$$

for all i. This means that the pair $(H^{-T}PH^{-1}, H^{-T}QH^{-1})$ is a common Riccati solution for the family $\mathcal{V}$.

Theorem 7. *For $i = 1, \ldots, m$, suppose that $A_i, B_i \in \mathbb{R}^{n\times n}$ provided that B_i have full rank for all i. If the family $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^m$ has common Riccati stability, then the family $\mathcal{V} = \{(A_i^T, B_i^T)\}_{i=1}^m$ has common Riccati stability*

Proof. Let (P, Q) be the common Riccati solution for $\mathcal{U}$, i.e., there are positive definte $R_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, such that

$$A_i^TP + PA_i + Q + PB_iQ^{-1}B_i^TP + R_i = 0.$$

Since, $P \succ 0$, then P^{-1} exists. Thus, by pre and and post-multiply these last equations, we obtain

$$P^{-1}A_i^T + A_iP^{-1} + P^{-1}QP^{-1} + B_iQ^{-1}B_i^T + P^{-1}R_iP^{-1} = 0.$$

Define

$$\hat{P} = P^{-1} \succ 0,$$
$$\hat{Q}_i = B_iQ^{-1}B_i^T \succ 0$$

and

$$\hat{R}_i = P^{-1}R_iP^{-1} \succ 0,$$

$i = 1, \ldots, m$. Since each B_i has a full rank, then B_i^{-1}, $i = 1, \ldots, m$, exists. Therefore, $\hat{Q}_i^{-1}$'s are defined. Thus, for $i = 1, \ldots, m$, the pair $(\hat{P}, \hat{Q}_i)$ is a Riccati solution for (A_i^T, B_i^T). To see this, for each i, observe that

$$(A_i^T)^T\hat{P} + \hat{P}A_i^T + \hat{Q}_i + \hat{P}B_i^T\hat{Q}_i^{-1}(B_i^T)^T\hat{P} + \hat{R}_i \\ = P^{-1}A_i^T + A_iP^{-1} + P^{-1}QP^{-1} + B_iQ^{-1}B_i^T + P^{-1}R_iP^{-1}.$$

The results presented in this section demonstrate that common Riccati stability is preserved under scaling transformations and similarity transformations. These findings underscore the robustness of Riccati-based stability criteria, which remain unaffected by coordinate transformations or uniform scaling of system parameters. By establishing that the Riccati solution remains valid across such transformations, we show that the proposed framework is not only theoretically sound but also practically applicable to a wide range of control systems, including those subject to rescaling or changes in system representation.

4. Examples and Computational Verification

In this section, we provide numerical examples to demonstrate the practical applicability of the proposed stability criteria.

Example 1. *Consider the family* $\mathcal{U} = \{(A_i, B_i)\}_{i=1}^2$, *where*

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.003 \end{bmatrix}, \; \text{and } B_2 = \begin{bmatrix} 0.02 & -0.019 \\ 0 & 0.23 \end{bmatrix}.$$

It can be easily verified that

$$P = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.1 & \\ & 0.1 \end{bmatrix}$$

form a common Riccati solution for $\mathcal{U}$. *A simple calculation shows that* P *is a common Lyapunov solution for the family* $\mathcal{A} = \{A_i\}_{i=1}^2$ *and* Q^{-1} *is a common Schur solution for the family* $\{A_i^{-1}B_i\}_{i=1}^2$, *verifying Theorems 1 and 2.*

Example 2. *Consider the family given in Example 1. A simple calculation shows that the matrix* P *and* $\bar{Q} = \begin{bmatrix} 0.1\beta & \\ & 0.1\beta \end{bmatrix}$ *form a common Riccati solution for the family* $\mathcal{V} = \{(\beta A_i, \beta B_i)\}_{i=1}^2$ *for any* $\beta > 0$, *as asserted by Theorem 5.*

Example 3. *Consider the nonsingular matrix* $H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, *and the family* $\mathcal{U}$ *from Example 1. A straightforward calculation shows that the pair*

$$P_1 = H^{-T} P H^{-1} = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \succ 0$$

and

$$Q_1 = H^{-T} Q H^{-1} = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.2 \end{bmatrix} \succ 0$$

form a common Riccati solution for the transformed family $\mathcal{V} = \{(HA_iH^{-1}, HB_iH^{-1})\}_{i=1}^2$. *This result is consistent with Theorem 6, which asserts that common Riccati stability is preserved under similarity transformations.*

5. Conclusion

In this work, we have explored fundamental stability properties of matrix families, establishing significant links between common Riccati stability, Lyapunov stability, and Schur stability. By demonstrating that common Riccati stability implies both Lyapunov and Schur stability, we have highlighted the utility of Riccati stability as a unifying concept in stability analysis. Additionally, we showed that common Riccati stability is preserved under scaling and similarity transformations, reinforcing its robustness in various applications. These results provide a foundation for future work in stability analysis and control theory, particularly in systems with time delays or those requiring stability across multiple configurations. This study's findings contribute valuable insights for the design of robust control systems that require stability despite parameter variations and transformations.

Acknowledgements

The authors gratefully acknowledge Qassim University, represented by the Deanship of Graduate Studies and Scientific Research, on the financial support for this research under the number (QU-J-UG-2-2025-56216) during the academic year 1446 AH / 2024 AD.

We also sincerely thank the reviewers for their valuable feedback and constructive comments, which have significantly contributed to enhancing the quality of this paper.

References

[1] R Horn and C Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[2] Lloyd N Trefethen and David Bau. *Numerical linear algebra.* SIAM, 2022.

[3] Roger A Horn and Charles R Johnson. *Topics in matrix analysis.* Cambridge university press, 1991.

[4] Eugenius Kaszkurewicz and Amit Bhaya. *Matrix diagonal stability in systems and computation.* Springer Science & Business Media, 2012.

[5] Robert K Brayton and Charles C Conley. Some results on the stability and instability of the backward differentiation methods with non-uniform time steps. *Topics in numerical analysis*, pages 13–33, 1972.

[6] Robert Brayton and Christopher Tong. Stability of dynamical systems: A constructive approach. *IEEE Transactions on Circuits and Systems*, 26(4):224–234, 1979.

[7] Stephen Boyd and Qinping Yang. Structured and simultaneous lyapunov functions for system stability problems. *International journal of Control*, 49(6):2215–2240, 1989.

[8] M Vidyasagar. On matrix measures and convex liapunov functions. *Journal of Mathematical Analysis and Applications*, 62(1):90–103, 1978.

[9] David Angeli, Nikolaos Athanasopoulos, Raphaël M Jungers, and Matthew Philippe. Path-complete graphs and common lyapunov functions. In *Proceedings of the 20th international conference on hybrid systems: computation and control*, pages 81–90, 2017.

[10] Stefania Andersen, Peter Giesl, and Sigurdur Hafstein. Common lyapunov functions for switched linear systems: Linear programming-based approach. *IEEE Control Systems Letters*, 7:901–906, 2022.

[11] GW Cross. Three types of matrix stability. *Linear algebra and its applications*, 20(3):253–263, 1978.

[12] Daniel Hershkowitz. Recent directions in matrix stability. *Linear Algebra and its Applications*, 171:161–186, 1992.

[13] Abraham Berman, Christopher King, and Robert Shorten. A characterisation of common diagonal stability over cones. *Linear and Multilinear Algebra*, 60(10):1117–1123, 2012.

[14] Ali Algefary. A characterization of common lyapunov diagonal stability using khatri-rao products. *AIMS Mathematics*, 9(8):20612–20626, 2024.

[15] Alan V Oppenheim. *Discrete-time signal processing*. Pearson Education India, 1999.

[16] Emilia Fridman. Introduction to time-delay systems. *Analysis and Control. Birkhäuser*, 75, 2014.

[17] Erik I Verliest and Anatoli F Ivanov. Robust stability of systems with delayed feedback. *Circuits, Systems and Signal Processing*, 13:213–222, 1994.

[18] Emilia Fridman. Tutorial on lyapunov-based methods for time-delay systems. *European Journal of Control*, 20(6):271–283, 2014.

[19] Vincent D Blondel and Alexandre Megretski. *Unsolved problems in mathematical systems and control theory*. Princeton University Press, 2009.

[20] Erik I Verriest. Robust stability and stabilization: from linear to nonlinear. *IFAC Proceedings Volumes*, 33(23):21–32, 2000.