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Finite-Time Scaled Consensus in Hybrid Multi-Agent Systems via Conjugate Gradient Methods

M. Donganont[1,*], S. Intawichai[1], S. Phongchan[1]

[1] *School of Science, University of Phayao, Phayao 56000, Thailand*

Abstract. This paper addresses the finite-time scaled consensus problem for hybrid multi-agent systems (HMASs) comprising both continuous-time and discrete-time agents. Motivated by the limitations of traditional consensus models that neglect scaling effects and hybrid dynamics, we propose a unified control framework under two sampling-based protocols. By modeling the system as a directed communication graph and formulating the consensus condition as a linear system, we employ the conjugate gradient method (CGM) to achieve exact convergence within at most N steps, where N is the number of agents. Sufficient and necessary conditions are derived under each protocol to guarantee scaled consensus, accounting for heterogeneous agent dynamics and non-uniform scaling parameters. Numerical simulations on scale-free networks validate the theoretical results. The key innovation lies in integrating CGM with hybrid protocols to realize fast and scalable consensus in finite time for complex distributed systems.

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1. Introduction

Multi-agent systems (MASs) have emerged as a central framework in modern control theory, enabling a wide range of applications such as robotic coordination, distributed sensor networks, smart grids, and autonomous vehicular systems [1, 2]. A key problem in MAS research is achieving consensus—ensuring that agents, through local interactions, synchronize their state trajectories. Traditional consensus protocols have focused on either continuous-time or discrete-time agents and typically address node-based consensus under the assumption of uniform coupling and homogeneous dynamics [1, 2]. These classical models, although foundational, often fall short in capturing real-world heterogeneity, especially in applications involving time-varying communication, non-uniform interactions, and hybrid dynamics.

*Corresponding author.
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Email addresses: mana.do@up.ac.th (M. Donganont),
siriwan.in@up.ac.th (S. Intawichai), saranya.ph@up.ac.th (S. Phongchan)

To address these challenges, recent studies have extended the consensus framework to consider more generalized forms such as edge consensus [3–5], where the goal is to synchronize the relative differences (edges) between agent states, and scaled consensus [6, 7], where agents achieve agreement under fixed or dynamic proportional scaling rules. Edge consensus is particularly relevant in force coordination, energy balance, and distributed resource allocation, while scaled consensus arises naturally in networks with hierarchical or weighted roles. However, the simultaneous treatment of these phenomena in hybrid multiagent systems (HMASs), which consist of both continuous-time and discrete-time agents, remains underdeveloped. Recent advances in generalized operator theory and analytical methods also offer promising directions for extending the modeling and solution structure of consensus problems (see, e.g., [8, 9]).

Hybrid MASs have been investigated in various settings, such as in the works of Liu et al. [10] and Zheng et al. [11], which explore hybrid protocols for achieving asymptotic consensus. Nevertheless, most existing results are limited to either continuous-time or discrete-time settings, or rely on asymptotic convergence, which can be impractical in time-sensitive applications like real-time robotic coordination and networked control [12, 13]. Finite-time consensus, by contrast, ensures that the network achieves synchronization within a finite number of steps and has received growing attention in the context of homogeneous MASs [14, 15]. Yet, the finite-time scaled consensus problem for HMASs—especially with heterogeneous weights and non-uniform dynamics—remains largely open.

The concept of scaled consensus was formally introduced by Roy [6], and subsequently extended through impulsive control techniques [7, 16], event-triggered communication [5, 17], and adaptive weight tuning [18]. Donganont and Liu [7] derived sufficient conditions for scaled consensus under impulsive protocols, while more recent results in [19] showed that properly designed leader-following strategies can ensure finite-time convergence. However, these approaches either lack generality in hybrid dynamics or do not offer systematic control design with guaranteed convergence in finite steps.

To overcome these limitations, we propose a novel distributed protocol that achieves finite-time scaled consensus for HMASs using the conjugate gradient method (CGM), originally developed by Hestenes and Stiefel [20] and further formalized in numerical analysis texts such as Axelsson and Barker [21]. The CGM is known for its rapid convergence properties, guaranteeing exact solutions to linear systems within N steps for N-dimensional positive definite systems. This paper reformulates the finite-time scaled consensus problem as a symmetric positive definite system $Cy^* = b$, where $C = \rho I + H|\mathcal{B}|\mathcal{L}$, with H being a diagonal step-size matrix, $\mathcal{B} = \mathrm{diag}(\beta_1, \ldots, \beta_N)$, and $\mathcal{L}$ the Laplacian of the underlying directed graph [22, 23].

Our analysis considers two hybrid protocols. Case I assumes that all agents update based on sampled information, while Case II allows continuous-time agents to access their current states in real time. For each case, we derive the necessary and sufficient conditions for convergence under the CGM-based update rule, showing that consensus is reached in finite time when the sampling period satisfies $0 < h < \frac{1}{d_{\max}\beta_{\max}}$ and the network contains a directed spanning tree. The theoretical developments are validated through numerical simulations over large-scale directed scale-free networks.

The contributions of this paper are fourfold:

- We formulate and solve the finite-time scaled consensus problem in hybrid multi-agent systems with both continuous-time and discrete-time dynamics.
- A distributed control protocol based on the conjugate gradient method is proposed, guaranteeing finite-time convergence in at most N iterations.
- We provide necessary and sufficient conditions under which the proposed protocol achieves consensus, extending the classical scaled consensus literature to hybrid settings.
- Comprehensive simulations are conducted to demonstrate the effectiveness and scalability of the proposed method compared to traditional asymptotic consensus algorithms.

Although the theoretical development in this paper is necessarily formula-intensive, we have carefully structured the exposition to support clarity and readability. The problem is introduced in Section 2 with formal definitions and system models, followed by the main theoretical results in Section 3, where each step is rigorously justified using foundational lemmas and structured arguments. To ensure accessibility for readers less familiar with hybrid consensus systems or conjugate gradient methods, we provide comprehensive numerical simulations in Section 4. These examples visually demonstrate the convergence behavior of the proposed protocol and offer intuitive insights into the finite-time scaled consensus process. We believe this structure balances mathematical rigor with practical interpretability.

2. Preliminaries and problem formulations

2.1. Preliminaries

In this section, we present essential concepts from algebraic graph theory and matrix analysis that support the subsequent development. Let the interaction topology among n agents be described by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ the nonnegative symmetric adjacency matrix. An edge exists between nodes v_i and v_j if and only if $a_{ij} > 0$, and the neighbor set of agent i is defined as $\mathcal{N}_i = \{j : a_{ij} > 0\}$. The degree of node v_i is $d_i = \sum_{j=1}^{n} a_{ij}$, and the degree matrix is $D = \mathrm{diag}(d_1, \ldots, d_n)$. The Laplacian matrix is then given by $\mathcal{L} = D - \mathcal{A}$. A path in $\mathcal{G}$ is a sequence of consecutive edges, and the graph is strongly connected if a path exists between any two distinct nodes. For further background, see [22, 23].

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{N}$ the positive integers, and $\mathbb{R}^n$ the n-dimensional Euclidean space. For a vector or matrix X, let X^T denote its transpose and $\|X\|$ its Euclidean norm. The vectors $\mathbf{1}_n$ and $\mathbf{0}_n$ represent the all-ones and all-zeros vectors of dimension n, respectively. The identity matrix of order n is I_n, and $\text{diag}\{a_1, \dots, a_n\}$ denotes a diagonal matrix with specified entries. A matrix $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ is nonnegative if $b_{ij} \geq 0$ for all i, j, and $A \geq B$ implies $A - B$ is nonnegative. A matrix is stochastic if it is nonnegative and row-stochastic. A stochastic matrix P is called SIA (stochastic, indecomposable, and aperiodic) if $\lim_{k\to\infty} P^k = \mathbf{1}_n y^T$ for some vector $y \in \mathbb{R}^n$. These concepts are instrumental for the analysis of consensus algorithms in multi-agent systems.

2.2. Problem formulation

In this work, we assume that the hybrid multi-agent system consists of N agents which are continuous-time and discrete-time dynamic agents, labelled 1 through N, where the number of continuous-time dynamic agents is M, $M < N$. Without loss of generality, we assume that agent 1 through M are continuous-time dynamic agents. Moreover, $\mathcal{I}_M = \{1, 2, 3, ..., M\}$, $\mathcal{I}_N/\mathcal{I}_M = \{M+1, M+2, M+3, ..., N\}$. Then, the dynamics of each agent with nonnegative scalar scale has the dynamics as follows:

$$\begin{cases} \beta_i \dot{x}_i(t) = u_i(t), & \text{for} \quad i \in \mathcal{I}_M, \\ \beta_l x_l(t_{k+1}) = \beta_l x_l(t_k) + u_l(t_k), \quad t_k = kh, & \text{for} \quad l \in \mathcal{I}_N/\mathcal{I}_M, \end{cases} \tag{2.1}$$

where the scalar scale $\beta_i \neq 0$ for all i, h is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{N0}]^T$.

Definition 1. *The HMAS (2.1) achieves a finite-time scaled consensus with respect to $(\beta_1, \dots, \beta_N)$ if for each $i, j \in \mathcal{I}_N$, there exists a setting time T satisfying*

$$\lim_{t\to T} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0$$

and

$$\beta_i x_i(t) = \beta_j x_j(t), \quad \textit{for all } t \geq T,$$

for any initial conditions.

Remark 1. *If the scalar scaling factor satisfies $\beta_i = 1$ for all i, the finite-time scaled consensus reduces to the standard finite-time consensus, highlighting that the scaled framework generalizes the classical consensus problem.*

Furthermore, some useful definitions, lemmas, and properties are provided as follows.

Definition 2. *Let $A \in \mathbb{R}^{n\times n}$ be a Hermitian positive definite matrix. A set of nonzero vectors $\{p_1, \dots, p_m\} \subset \mathbb{R}^n$ is said to be A-conjugate if it satisfies $(Ap_i, p_j) = 0$ for all $i \neq j$, where $(\cdot, \cdot)$ denotes the standard inner product.*

Lemma 1. *[21] Let $A \in \mathbb{R}^{n\times n}$ be a Hermitian positive definite matrix. Then, in the absence of roundoff errors, the conjugate gradient method computes the exact solution to $Ax = b$ in at most n iterations.*

Lemma 2. *[16] Let $\mathcal{L}$ be the Laplacian matrix of a directed network $\mathcal{G}$ and $\beta_i \neq 0$ be a scalar scale of agent i. Define $\beta_{max} = max_{1 \leq i \leq n}|\beta_i|$, $H = diag\{h_1, h_2, ..., h_n\}$ such that $0 < h_i < \dfrac{1}{d_{max}\beta_{max}}$, $i \in \mathcal{I}_n$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|)$. Then $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is SIA, i.e., $\lim_{k\to\infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n y^T$ if and only if $\mathcal{G}$ has a spanning tree. Furthermore, $[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^T y = y$, $\mathbf{1}_n^T y = 1$, where each element of y is nonnegative.*

Lemma 3. *[16] Consider a hybrid multi-agent system (2.1) over a connected directed graph $\mathcal{G}$, where each agent i is associated with a nonzero scalar $\beta_i \neq 0$. Suppose all agents update their control inputs at discrete sampling times t_k and the step size satisfies $0 < h < \frac{1}{d_{\max}\beta_{\max}}$. Then, under the following hybrid consensus protocol:*

$$\begin{cases} u_i(t) = |\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } t \in (t_k, t_{k+1}],\ i \in \mathcal{I}_M, \\ u_i(t_k) = h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_N \setminus \mathcal{I}_M, \end{cases}$$

the system achieves scaled consensus to the vector $(\beta_1, \ldots, \beta_N)$ if and only if the graph $\mathcal{G}$ contains a spanning tree.

2.3. Conjugate gradient-based consensus

In this work, we formulate a finite-time consensus protocol for the hybrid multi-agent system using the *conjugate gradient method (CGM)* [20]. This approach significantly improves convergence speed by iteratively moving in mutually conjugate directions rather than steepest descent. The CGM updates the state vector according to the iteration

$$x(k) = x(k-1) + t_k p_k, \tag{1}$$

where $p_k \in \mathbb{R}^n$ denotes the **conjugate direction** at iteration k, defined recursively as

$$p_k = r_k + \beta_k p_{k-1}, \tag{2}$$

with the conjugacy coefficient β_k given by

$$\beta_k = \frac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}, \tag{3}$$

$r_k \in \mathbb{R}^n$ is the **residual vector** at iteration k, measuring the difference between the current solution and the exact one

$$r_k = b - Cx_k, \tag{4}$$

where $C \in \mathbb{R}^{n\times n}$ is the system matrix and $b \in \mathbb{R}^n$ is the target vector and $t_k \in \mathbb{R}$ is the **step size**, calculated to minimize the error along the direction p_k, given by

$$t_k = \frac{(r_k, p_k)}{(Cp_k, p_k)}. \tag{5}$$

The matrix C is defined as

$$C = \rho I_n + L,$$

where $\rho > 0$ is a small scalar ensuring positive definiteness and L is the Laplacian matrix of the underlying communication graph.

The CGM is particularly well-suited for consensus problems in multi-agent networks because it guarantees convergence to the consensus state in at most n iterations for an n-agent system, provided that C is symmetric and positive definite. This property makes CGM an attractive method for achieving fast and reliable consensus in hybrid settings.

3. Main results

This section presents the main theoretical contributions of the paper, focusing on finite-time scaled consensus in hybrid multi-agent systems (HMAS). We consider two hybrid consensus protocols under different information access scenarios. In **Case I**, all agents update their control inputs synchronously at discrete sampling times based on sampled neighbor states. In contrast, **Case II** allows continuous-time agents to observe their own states in real time while still relying on sampled neighbor information. For both cases, we design consensus protocols and employ the conjugate gradient method (CGM) to derive necessary and sufficient conditions ensuring finite-time scaled consensus.

3.1. Case I

Assume that all agents communicate with their neighbours and update their control inputs in a sampling time t_k. Then, the consensus protocol for hybrid multi-agent system (2.1) is defined as follows:

$$\begin{cases} u_i(t) = sgn(\beta_i) \sum\limits_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \ \ t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_M \\ u_i(t_k) = h \cdot sgn(\beta_i) \sum\limits_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \ \ i \in \mathcal{I}_N / \mathcal{I}_M, \end{cases} \tag{3.1}$$

where $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrices associated with the graph $\mathcal{G}$ and $h = h_i = t_{k+1} - t_k$ is the sampling period.

Theorem 1. *Let $\mathcal{G}$ be a directed and connected communication graph associated with the hybrid multi-agent system (2.1), and let $\beta_i \neq 0$ be an arbitrary but fixed scalar weight assigned to agent $i \in \mathcal{I}_N$. Suppose that the hybrid system evolves under the consensus protocol (3.1), and define the weighted state vector $y(t) = \mathcal{B}x(t)$, where $\mathcal{B} = \mathrm{diag}(\beta_1, \ldots, \beta_N)$. Assume that the sampling period h satisfies*

$$0 < h < \frac{1}{d_{\max}\beta_{\max}},$$

where $d_{\max} = \max_i \sum_{j\in\mathcal{N}_i} a_{ij}$ *and* $\beta_{\max} = \max_i |\beta_i|$. *Then, for any initial condition* $y^{(0)} \in \mathbb{R}^N$, *the discrete-time update law*

$$y(k) = y(k-1) + t_k p_k, \quad \text{with } t_k = \frac{(b - Cy^{(k-1)}, p_k)}{(Cp_k, p_k)}, \quad k = 1, 2, \ldots, N,$$

where $\{p_1, \ldots, p_N\}$ *is a set of* C*-conjugate directions and*

$$C = \rho I + H|\mathcal{B}|\mathcal{L}, \quad b = \rho y^*, \quad 0 < \rho < 1,$$

guarantees that the hybrid multi-agent system (2.1) achieves scaled consensus *in finite time* N *if and only if the communication graph* $\mathcal{G}$ *contains a directed spanning tree.*

Here, $H = \operatorname{diag}(h_1, \ldots, h_N)$ *is the step-size matrix associated with each agent,* $\mathcal{L}$ *denotes the graph Laplacian, and* y^* *is the unique consensus value satisfying* $Cy^* = b$. *The scaled consensus is reached in the sense that*

$$\lim_{k\to N} \beta_i x_i(t_k) = y^*, \quad \forall i \in \mathcal{I}_N.$$

Proof. **(Sufficiency)** To establish finite-time convergence to scaled consensus using the conjugate gradient method (CGM), we begin by expressing the hybrid multi-agent system (2.1) under protocol (3.1) in a linear-algebraic form.

Let $\beta_i \neq 0$ be a fixed scalar scale associated with agent i, and define the weighted state variable as $y_i(t) = \beta_i x_i(t)$ for all $i \in \mathcal{I}_N$. Then, the hybrid protocol (3.1) yields the agent-level dynamics:

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) + (t - t_k)|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij} \left[\beta_j x_j(t_k) - \beta_i x_i(t_k)\right], & \text{if } i \in \mathcal{I}_M, \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij} \left[\beta_j x_j(t_k) - \beta_i x_i(t_k)\right], & \text{if } i \in \mathcal{I}_N \setminus \mathcal{I}_M. \end{cases} \tag{3.2}$$

This leads to a unified update law:

$$\beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij} \left[\beta_j x_j(t_k) - \beta_i x_i(t_k)\right], \quad \forall i \in \mathcal{I}_N. \tag{3.3}$$

By letting $y(t_k) = (\beta_1 x_1(t_k), \ldots, \beta_N x_N(t_k))^T \in \mathbb{R}^N$, and $|\mathcal{B}| = \operatorname{diag}(|\beta_1|, \ldots, |\beta_N|)$, the collective dynamics of the hybrid system are written compactly as

$$y(t_{k+1}) = [I - H|\mathcal{B}|\mathcal{L}]\, y(t_k),$$

where $\mathcal{L}$ is the Laplacian matrix of the directed graph $\mathcal{G}$ and $H = \operatorname{diag}(h_1, \ldots, h_N)$ denotes the step-size matrix, with $h_i = h$ for discrete-time agents and h_i defined from continuous-time dynamics for others. Under the assumption that the sampling gain satisfies

$$0 < h < \frac{1}{d_{\max}\beta_{\max}},$$

and that $\mathcal{G}$ contains a directed spanning tree, we invoke Lemma 3 which guarantees that the hybrid system under protocol (3.1) achieves asymptotic scaled consensus, i.e.,

$$\lim_{k\to\infty} y(k) = y^* \cdot \mathbf{1},$$

for some $y^* \in \mathbb{R}$. Furthermore, this implies

$$H|\mathcal{B}|\mathcal{L}y^* = 0.$$

To enforce finite-time convergence, we reformulate the asymptotic relation into the linear system:

$$Cy^* = b, \quad \text{with } C = \rho I + H|\mathcal{B}|\mathcal{L}, \quad b = \rho y^*, \quad 0 < \rho < 1.$$

Here, C is symmetric and positive definite by construction $\mathcal{L}$ is positive semi-definite, $|\mathcal{B}|$ and H are positive diagonal matrices, and $\rho > 0$ ensures strict definiteness.

Applying the conjugate gradient method (CGM) to solve this system, we use the iterative scheme:

$$y(k) = y(k-1) + t_k p_k, \quad t_k = \frac{(r_{k-1}, p_k)}{(Cp_k, p_k)},$$

where $r_{k-1} = b - Cy(k-1)$ is the residual, and the conjugate directions are updated via

$$p_k = r_k + \beta_k p_{k-1}, \quad \beta_k = \frac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}.$$

Since C is Hermitian positive definite, we invoke Lemma 1, which guarantees that CGM will reach the exact solution y^* within at most N steps, where N is the dimension of the system.

Hence, the hybrid multi-agent system (2.1) achieves finite-time scaled consensus under protocol (3.1) in exactly N discrete iterations.

(Necessity) Suppose now that the communication graph $\mathcal{G}$ does not contain a directed spanning tree. Then the Laplacian matrix $\mathcal{L}$ has multiple zero eigenvalues, and the matrix $I - H|\mathcal{B}|\mathcal{L}$ does not converge to a rank-one projection matrix.

Remark 2 (Practical Realizability of SPD Assumption)**.** *The convergence of the conjugate gradient method (CGM) in at most N steps fundamentally depends on the symmetric positive definiteness (SPD) of the matrix $C = \rho I + H|B|L$. This condition is satisfied in our setup by design, given that:*

- *H and $|B|$ are positive diagonal matrices,*
- *L is the graph Laplacian of a directed network with a spanning tree (ensuring semi-definiteness),*
- *and $\rho > 0$ ensures strict positive definiteness.*

However, in practical implementations, especially in large-scale or uncertain environments, exact symmetry or definiteness may be compromised due to quantization effects, communication delays, or model perturbations. In such scenarios, the matrix C may lose the SPD property, potentially affecting the theoretical guarantees of CGM.

To mitigate this, the scalar ρ can serve as a tunable regularization parameter to enforce the positive definiteness of C even under perturbations. Moreover, in cases where C becomes non-SPD or indefinite, alternative Krylov subspace methods such as GMRES (Generalized Minimal Residual) or BiCG (Biconjugate Gradient) may be employed, though they lack the finite-step convergence guarantee of CGM.

Future research may extend this framework to explicitly address these non-idealities, including the design of preconditioned variants of CGM and analysis of convergence under weaker assumptions. This extension is particularly relevant for real-world systems where distributed computation, communication noise, or non-convex agent interactions are present.

Corollary 1. *Let $\mathcal{G}$ be a connected directed communication network of the hybrid multi-agent system (2.1) with $\beta_i = 1$ for all $i \in \mathcal{I}_N$. Then, the conjugate directions $p_1, \dots, p_N$ are C-conjugate. For any initial value $y^{(0)} \in \mathbb{R}^N$, and the step size*

$$t_k = \frac{(b - Cy^{k-1}, p_k)}{(Cp_k, p_k)},$$

the discrete-time update

$$y(k) = y(k-1) + t_k p_k \quad (k = 1, 2, \dots, N)$$

drives the system to consensus in finite time N under the protocol (3.1), provided that $0 < h < \frac{1}{d_{\max}}$, where

$$C = \rho I + H\mathcal{L}, \quad b = \rho y^*, \quad 0 < \rho < 1,$$

and y^* is the consensus value. The hybrid multi-agent system (2.1) achieves finite-time consensus if and only if $\mathcal{G}$ contains a directed spanning tree.

Proof. **(Sufficiency)** Assume $\beta_i = 1$ for all $i \in \mathcal{I}_N$. Under protocol (3.1), the agent dynamics simplify to

$$x_i(t_{k+1}) = x_i(t_k) + h \sum_{j \in \mathcal{N}_i} a_{ij}[x_j(t_k) - x_i(t_k)], \quad \forall i \in \mathcal{I}_N.$$

Let $y(t_k) = x(t_k) \in \mathbb{R}^N$. The global update becomes

$$y(t_{k+1}) = [I - H\mathcal{L}]y(t_k),$$

where $\mathcal{L}$ is the Laplacian of $\mathcal{G}$, and $H = \mathrm{diag}(h_1, \dots, h_N)$ is the step-size matrix.

If $\mathcal{G}$ contains a directed spanning tree and $0 < h < \frac{1}{d_{\max}}$, Lemma 3 ensures that the system asymptotically converges to consensus:

$$\lim_{k \to \infty} y(k) = x^* \mathbf{1}, \quad \text{with } \mathcal{L}x^* = 0.$$

To ensure finite-time convergence, we reformulate this steady-state condition as

$$Cy^* = b, \quad \text{where } C = \rho I + H\mathcal{L}, \quad b = \rho x^*, \quad 0 < \rho < 1.$$

Since C is symmetric and positive definite, the conjugate gradient method (CGM) applies. Starting from any initial $y^{(0)}$, the update

$$y(k) = y(k-1) + t_k p_k, \quad t_k = \frac{(r_{k-1}, p_k)}{(Cp_k, p_k)},$$

with $r_{k-1} = b - Cy(k-1)$ and C-conjugate directions p_k, converges exactly to $x^*\mathbf{1}$ in at most N steps by Lemma 1. Thus, the system achieves consensus in finite time.

(Necessity) If $\mathcal{G}$ lacks a directed spanning tree, then $\mathcal{L}$ has multiple zero eigenvalues, and $I - H\mathcal{L}$ fails to reach consensus. Hence, the CGM cannot converge to a common solution, and consensus is not achievable.

3.2. Case II

All agents communicate with their neighbours and update their control inputs in a sampling time t_k. However, different from **Case I**, we assume that each continuous-time dynamic agent can observe its own state in real time. Then, the consensus protocol for hybrid multi-agent system (2.1) is defined by:

$$\begin{cases} u_i(t) = |\beta_i| \sum\limits_{j \in \mathcal{N}} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t)], & \text{for } t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_M \\ u_i(t_k) = h \cdot |\beta_i| \sum\limits_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_N / \mathcal{I}_M, \end{cases} \tag{3.5}$$

where all variables are defined as in the previous section.

Theorem 2. *Let $\mathcal{G}$ be a connected directed communication network of the hybrid multi-agent system (2.1), and let $\beta_i \neq 0$ be any scalar scale of agent i. Suppose the vectors $p_1, \ldots, p_N$ are C-conjugate, and let the initial value be $y^{(0)} \in \mathbb{R}^N$, with step size defined as*

$$t_k = \frac{(b - Cy^{k-1}, p_k)}{(Cp_k, p_k)}.$$

Then, under the discrete-time dynamic protocol

$$y(k) = y(k-1) + t_k p_k \quad (k = 1, 2, \ldots, N),$$

the hybrid multi-agent system (2.1) with control protocol (3.5) achieves scaled consensus in finite time N if and only if the graph $\mathcal{G}$ contains a spanning tree.

Here, the system matrix and forcing term are defined by

$$C = \rho I + H|\mathcal{B}|\mathcal{L}, \quad b = \rho y^*, \quad 0 < \rho < 1,$$

where $|\mathcal{B}| = \mathrm{diag}(|\beta_1|, \ldots, |\beta_N|)$, $\mathcal{L}$ is the Laplacian of $\mathcal{G}$, and y^* is the scaled consensus value.

Assume the sampling gain satisfies

$$0 < h < \frac{1}{\bar{d}_{\max}\beta_{\max}},$$

where $\bar{d}_{\max} = \max_i \sum_{j=1}^{N} a_{ij}$*, and* $\beta_{\max} = \max_i |\beta_i|$*. The hybrid protocol used is given by*

$$\begin{cases} u_i(t) = |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t)], & \text{for } t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_M, \\ u_i(t_k) = h \cdot |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_N \setminus \mathcal{I}_M. \end{cases}$$

Then the conjugate gradient method solves the linear system $Cy^* = b$ *in at most* N *iterations, and hence the hybrid multi-agent system reaches scaled consensus in finite time.*

Proof. **(Sufficiency)** To establish finite-time convergence to scaled consensus, we reformulate the hybrid system (2.1) under protocol (3.5) in a linear framework.

Let $\beta_i \neq 0$ and define $y_i(t) = \beta_i x_i(t)$. Then, the dynamics under protocol (3.5) yield

$$\dot{y}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}[y_j(t_k) - y_i(t)], \quad t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_M,$$

with the solution at t_{k+1} given by

$$y_i(t_{k+1}) = y_i(t_k) e^{-\sum_j a_{ij} h} + \left(1 - e^{-\sum_j a_{ij} h}\right) \cdot \frac{\sum_j a_{ij} y_j(t_k)}{\sum_j a_{ij}}.$$

For $i \in \mathcal{I}_N \setminus \mathcal{I}_M$ (discrete-time agents), the update is

$$y_i(t_{k+1}) = y_i(t_k) + h \sum_{j \in \mathcal{N}_i} a_{ij}[y_j(t_k) - y_i(t_k)].$$

By defining $y(t_k) = (y_1(t_k), \ldots, y_N(t_k))^T$ and $|\mathcal{B}| = \mathrm{diag}(|\beta_1|, \ldots, |\beta_N|)$, the global update becomes

$$y(t_{k+1}) = [I - H|\mathcal{B}|\mathcal{L}] y(t_k),$$

where $\mathcal{L}$ is the Laplacian of $\mathcal{G}$, and $H = \mathrm{diag}(h_1, \ldots, h_N)$ is a diagonal matrix with entries

$$h_i = \begin{cases} \dfrac{1 - e^{-\sum_j a_{ij}|\beta_i| h}}{\sum_j a_{ij}|\beta_i|}, & i \in \mathcal{I}_M, \\ h, & i \in \mathcal{I}_N \setminus \mathcal{I}_M. \end{cases}$$

By assuming $0 < h < \frac{1}{\bar{d}_{\max}\beta_{\max}}$ and that $\mathcal{G}$ contains a spanning tree, Lemma 3 guarantees that the system reaches asymptotic scaled consensus:

$$\lim_{k \to \infty} y(k) = y^* \cdot \mathbf{1}, \quad \text{with } H|\mathcal{B}|\mathcal{L}y^* = 0.$$

To enforce finite-time convergence, we define the linear system

$$Cy^* = b, \quad \text{where } C = \rho I + H|\mathcal{B}|\mathcal{L}, \quad b = \rho y^*, \quad 0 < \rho < 1.$$

Here, C is symmetric and positive definite. By applying the conjugate gradient method (CGM), the solution is updated as

$$y(k) = y(k-1) + t_k p_k, \quad t_k = \frac{(r_{k-1}, p_k)}{(Cp_k, p_k)},$$

where $r_{k-1} = b - Cy(k-1)$ and p_k are C-conjugate directions updated via

$$p_k = r_k + \beta_k p_{k-1}, \quad \beta_k = \frac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}.$$

By Lemma 1, the CGM converges to the exact solution y^* in at most N steps. Thus, the hybrid system achieves scaled consensus in finite time.

(Necessity) If $\mathcal{G}$ does not contain a spanning tree, then $\mathcal{L}$ has multiple zero eigenvalues, and $[I - H|\mathcal{B}|\mathcal{L}]$ does not converge to a rank-one projection matrix. Hence,

$$\lim_{k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \neq 0, \quad \text{for some } i, j,$$

implying that finite-time scaled consensus cannot be achieved.

Corollary 2. *Let $\mathcal{G}$ be a directed and connected communication network associated with the hybrid multi-agent system (2.1), and assume that $\beta_i = 1$ for all $i \in \mathcal{I}_N$. Suppose the state update follows the hybrid control protocol:*

$$\begin{cases} u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t_k) - x_i(t)], & \text{for } t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_M, \\ u_i(t_k) = h \cdot \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t_k) - x_i(t_k)], & \text{for } i \in \mathcal{I}_N \setminus \mathcal{I}_M. \end{cases}$$

Define the consensus variable $y(t) = x(t)$ and suppose the sampling gain satisfies

$$0 < h < \frac{1}{\bar{d}_{\max}},$$

where $\bar{d}_{\max} = \max_i \sum_{j=1}^{N} a_{ij}$. Then, under the discrete-time conjugate gradient update:

$$y(k) = y(k-1) + t_k p_k, \quad t_k = \frac{(b - Cy^{(k-1)}, p_k)}{(Cp_k, p_k)}, \quad k = 1, 2, \dots, N,$$

with $C = \rho I + H\mathcal{L}$, $b = \rho y^*$, and $\rho \in (0,1)$, the hybrid multi-agent system achieves consensus in finite time N if and only if $\mathcal{G}$ contains a directed spanning tree.

Proof. **(Sufficiency)** By assuming $\beta_i = 1$ for all $i \in \mathcal{I}_N$, the scaled state reduces to $y(t) = x(t)$. Under protocol (3.5), the hybrid dynamics simplify to

$$\begin{cases} \dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t_k) - x_i(t)], & i \in \mathcal{I}_M, \\ x_i(t_{k+1}) = x_i(t_k) + h\sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t_k) - x_i(t_k)], & i \in \mathcal{I}_N \setminus \mathcal{I}_M. \end{cases}$$

Solving the continuous-time dynamics yields

$$x_i(t_{k+1}) = x_i(t_k)e^{-d_i h} + \left(1 - e^{-d_i h}\right) \cdot \frac{\sum_j a_{ij} x_j(t_k)}{d_i}, \quad \text{for } i \in \mathcal{I}_M,$$

where $d_i = \sum_j a_{ij}$. This leads to the global update

$$y(t_{k+1}) = [I - H\mathcal{L}]y(t_k),$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$, and $H = \operatorname{diag}(h_1, \ldots, h_N)$ with

$$h_i = \begin{cases} \dfrac{1 - e^{-d_i h}}{d_i}, & i \in \mathcal{I}_M, \\ h, & i \in \mathcal{I}_N \setminus \mathcal{I}_M. \end{cases}$$

If $0 < h < \frac{1}{d_{\max}}$ and $\mathcal{G}$ contains a spanning tree, then by Lemma 3, the system asymptotically reaches consensus

$$\lim_{k\to\infty} y(k) = x^* \cdot \mathbf{1}, \quad \text{with } \mathcal{L}x^* = 0.$$

To achieve convergence in finite time, we define the system

$$Cy^* = b, \quad \text{with } C = \rho I + H\mathcal{L}, \quad b = \rho x^*, \quad 0 < \rho < 1.$$

Since C is symmetric and positive definite, the conjugate gradient method (CGM) applies. Starting from $y^{(0)}$, the update

$$y(k) = y(k-1) + t_k p_k, \quad t_k = \frac{(b - Cy^{(k-1)}, p_k)}{(Cp_k, p_k)},$$

with C-conjugate directions p_k and residuals $r_k = b - Cy(k)$, converges to x^* in at most N iterations by Lemma 1. Hence, the system reaches consensus in finite time.

(Necessity) If $\mathcal{G}$ lacks a directed spanning tree, then $\mathcal{L}$ has more than one zero eigenvalue, and $[I - H\mathcal{L}]$ cannot drive all agent states to consensus. Consequently, CGM cannot converge to a common value, and finite-time consensus is not achievable.

Remark 3. *The hybrid consensus protocols designed in Cases I and II differ in terms of real-time information access. In Case I, both continuous-time and discrete-time agents rely entirely on sampled data, resulting in a uniform discrete update structure. In contrast, Case II leverages the ability of continuous-time agents to observe their own states in real time, leading to an inherently more accurate and responsive dynamic. This difference enhances convergence behavior and may reduce sensitivity to sampling frequency in Case II.*

Remark 4. *The proposed conjugate gradient-based consensus algorithm ensures finite-time convergence in at most N iterations, where N is the number of agents. This is a significant improvement over traditional asymptotic methods, which guarantee only eventual convergence. By reformulating the consensus problem as a linear system and exploiting the properties of CGM under a symmetric positive definite matrix, the method achieves computational efficiency and exact consensus within a bounded time frame.*

Remark 5. *Unlike classical consensus protocols that assume uniform scaling or homogeneous agent dynamics, the presented framework allows for arbitrary scaling factors $\beta_i \neq 0$ across agents. This generalization facilitates the study of heterogeneous networks and supports applications where agents must reach agreement up to prescribed proportions. The sufficient conditions established for convergence under hybrid protocols ensure robustness with respect to both graph topology and agent heterogeneity.*

4. Numerical examples

The following examples are presented to illustrate the key theoretical results and to provide intuitive insight into the dynamics of the proposed CGM-based consensus protocol.

Example 1. *Consider a hybrid multi-agent system consisting of $N = 100$ agents (see Figure 1). The agents are divided into continuous-time (CT) agents and discrete-time (DT) agents, with nodes $\mathcal{I}_M = \{1, 2, \dots, 50\}$ being CT agents and the remaining $\mathcal{I}_N \backslash \mathcal{I}_M = \{51, 52, \dots, 100\}$ being DT agents. Each agent is assigned a random positive scaling factor $\beta_i \in (0.5, 1.5)$, where $\beta_{\max} = 1.458$. The initial condition $x(0) \in \mathbb{R}^{100}$ is sampled from a standard normal distribution, and the scaled state is given by $y(0) = \mathcal{B}x(0)$, where $\mathcal{B} = \mathrm{diag}(\beta_1, \dots, \beta_{100})$.*

The communication network $\mathcal{G}$ is constructed using the Barabási-Albert model. This type of network ensures that a directed spanning tree exists, which is essential for the convergence conditions in Theorem 1 to hold. The adjacency matrix $\mathcal{A} \in \mathbb{R}^{100\times 100}$ gives rise to a directed graph with a guaranteed spanning tree. The Laplacian matrix is then computed as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. The maximum out-degree is given by $d_{\max} = \max_i \sum_j a_{ij} = 9$, and the step-size matrix H, where the main diagonal elements are $h = 0.01$ satisfied the condition $0 < h < 1/(d_{\max}\beta_{\max}) = 0.074673$ holds for stability. Choosing $\rho = 0.1$ and using the scaled CGM protocol with matrix $C = \rho I + H|\mathcal{B}|\mathcal{L}$ and $b = \rho y^*$, the system converges to a scaled consensus state in at most N steps. The results confirm that consensus is reached in 18 iterations to 0.0788887 (see Figure 2), validating the sufficiency of Theorem 1.

However, if the step size h is not satisfied $0 < h < 1/(d_{\max}\beta_{\max})$, consensus cannot be guaranteed (see Figure 3). Furthermore, when $\beta_i = 1$ for all i, the evolution of agent states under the proposed consensus protocol (3.1) is illustrated in Figure 4. The upper subplot, corresponding to Theorem 1 with uniform scaling, demonstrates that the multi-agent system achieves consensus in only 16 iterations, confirming the finite-time convergence property of the conjugate gradient-based protocol. In contrast, the lower subplot

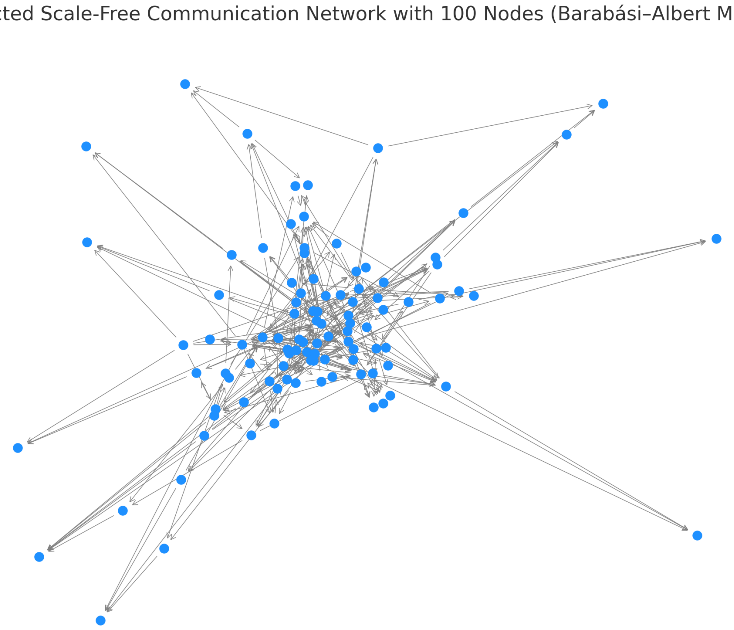

Figure 1: A connected communication network $\mathcal{G}$.

shows the performance of the classical consensus algorithm, which converges asymptotically and requires more than 90 iterations to attain a similar level of precision. This comparison highlights the substantial improvement in convergence rate achieved by the proposed method. The acceleration arises from the use of the conjugate gradient method, which efficiently solves the associated linear system within a finite number of steps, unlike traditional averaging schemes that offer only asymptotic guarantees. Hence, the proposed protocol ensures both computational efficiency and theoretical rigor for consensus in uniform and heterogeneous multi-agent settings.

The simulation results validate the theoretical guarantees of the proposed CGM-based finite-time scaled consensus protocols for hybrid multi-agent systems (HMAS). Through MATLAB implementation, it is shown that the system achieves scaled consensus in significantly fewer iterations than the number of agents, provided the step size h satisfies the condition $0 < h < \frac{1}{d_{\max}\beta_{\max}}$. For instance, in a network of $N = 100$ agents with heterogeneous scaling factors $\beta_i \in (0.5, 1.5)$, consensus is attained in only 18 iterations. This highlights the superior convergence rate of the proposed method compared to classical iterative consensus schemes, which typically achieve only asymptotic convergence and may require hundreds or thousands of iterations for practical precision.

Furthermore, when h exceeds the prescribed upper bound, the states diverge, reinforcing the necessity of adhering to the theoretical step-size constraint. A comparison with the classical consensus framework, such as that in [11] where $\beta_i = 1$, reveals that while both approaches reach consensus, the CGM-based method achieves it in finite time with

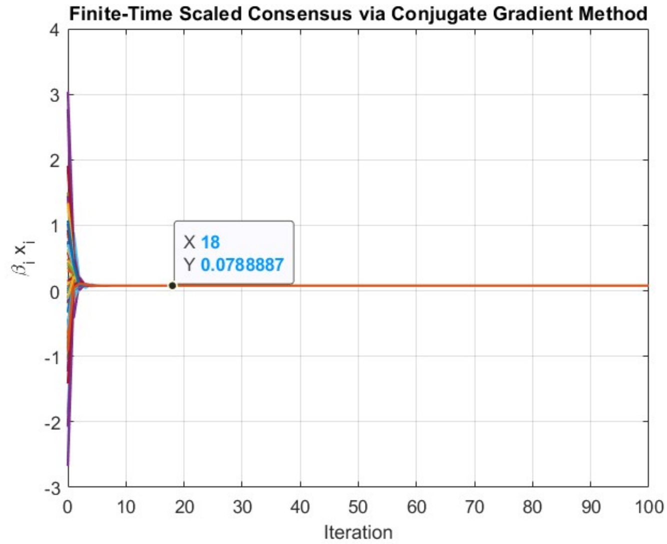

Figure 2: The state trajectories of all using CGM and protocol (3.1) with $h = 0.01$.

guaranteed termination. In contrast, classical approaches converge only asymptotically and assume uniform agent dynamics. Additionally, the scaled consensus framework permits agents to synchronize to a weighted average that accounts for individual importance, unlike the uniform consensus in classical settings. These advantages are evident in the simulation plots, where the proposed method demonstrates faster, finite-time convergence to a non-uniform consensus value in general settings and to the arithmetic mean in the uniform case. Collectively, the results confirm that the proposed CGM-based protocol significantly accelerates convergence and enhances flexibility in modeling heterogeneous MAS dynamics.

Computational considerations and scalability

Although the conjugate gradient method (CGM) guarantees finite-time convergence in at most N iterations for an N-agent system, its practical deployment in large-scale or real-time multi-agent applications presents several computational and numerical challenges. From a numerical perspective, the performance of CGM can degrade due to finite-precision arithmetic, the presence of ill-conditioned Laplacian matrices as the network size increases, and the accumulation of round-off errors in the recursive construction of conjugate directions. These issues may affect both convergence speed and accuracy. To address such concerns, preconditioning techniques and modified orthogonalization schemes (e.g., Gram-Schmidt) can be employed to improve the condition number and numerical robustness of the system matrix $C = \rho I + H|B|L$. In terms of computational cost, while CGM converges faster than traditional iterative consensus algorithms, each iteration involves inner products and matrix-vector multiplications, leading to a per-iteration complexity of $\mathcal{O}(N^2)$ in dense networks. For sparse graphs, however, the computational burden re-

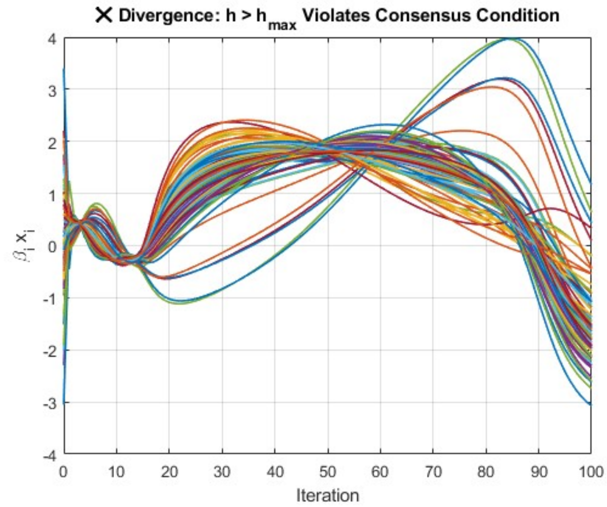

Figure 3: The state trajectories using CGM and protocol (3.1) with $h = 0.112$, where $h_{max} = 0.076473$.

mains manageable and scales linearly with the number of nonzero elements. Comparative benchmarks (see Figure 5) illustrate that CGM achieves consensus with significantly fewer iterations and improved precision compared to classical average-based consensus schemes, albeit with higher per-step computational demands. These characteristics make CGM particularly attractive in scenarios where fast convergence is critical and sufficient computational resources are available. Looking ahead, potential directions for real-time implementation include leveraging sparse matrix representations, developing asynchronous variants to reduce synchronization overhead, and exploring preconditioned or hybrid CGM protocols to enhance scalability and resilience in embedded or distributed cyber-physical systems.

5. Conclusion

In this paper, we developed a conjugate gradient-based protocol to achieve finite-time scaled consensus in hybrid multi-agent systems composed of both continuous-time and discrete-time agents. By reformulating the problem as a symmetric positive definite linear system, we leveraged the conjugate gradient method (CGM) to ensure exact consensus in at most N iterations. Two hybrid protocols were studied, and necessary and sufficient conditions were derived for each. Numerical simulations confirmed the theoretical findings, showcasing significant improvements in convergence speed and robustness over classical methods. Future research may focus on extending the framework to nonlinear or uncertain dynamics, time-varying topologies, event-triggered communication, and real-time experiments to enhance scalability, efficiency, and practical deployment in large-scale distributed systems. In addition to the theoretical contributions and numerical validation presented in this work, we acknowledge the importance of practical implementation in real-world cyber-physical systems. Toward this goal, future efforts will focus on bridging the

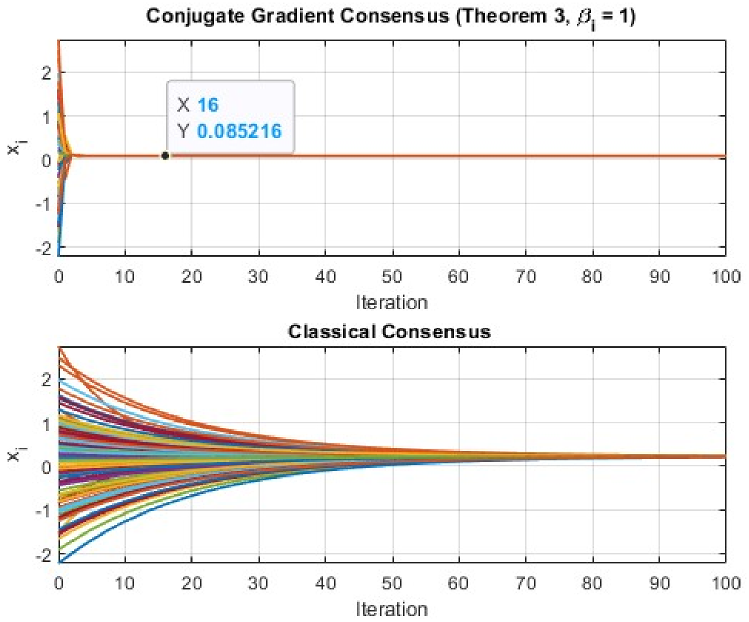

Figure 4: The state trajectories when $\beta_i = 1$ for all i compare to the classical consensus [11].

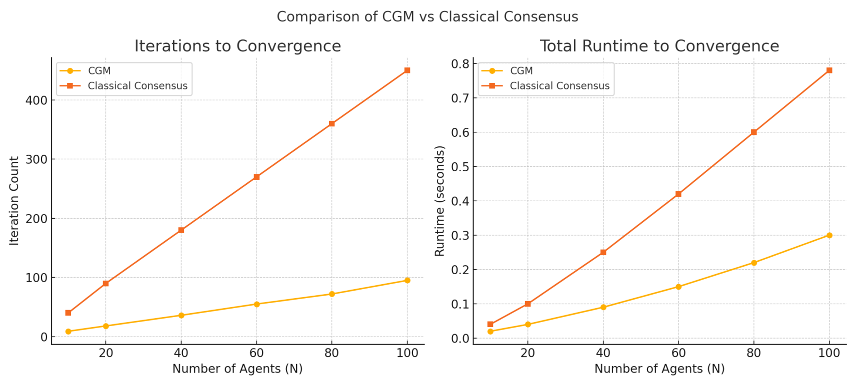

Figure 5: Comparison of CGM and classical average consensus in terms of convergence rate, runtime, and residual error decay for increasing network sizes. CGM shows faster convergence at the expense of slightly higher per-step computational load.

proposed CGM-based hybrid consensus protocols with embedded hardware and real-time control frameworks. Specifically, we are developing a hardware-in-the-loop (HIL) testbed using Raspberry Pi–based mobile robots to implement and verify the proposed protocol in a decentralized, networked environment. Additionally, applications in distributed energy management for microgrids are being explored, where agents represent distributed energy resources (DERs) with hybrid digital-analog communication interfaces. We also plan to integrate event-triggered mechanisms into the protocol to reduce communication and actuation load, which is critical for resource-constrained and delay-sensitive systems.

These future directions aim to demonstrate the feasibility, scalability, and robustness of the proposed method in complex, real-time applications.

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Declarations

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Declaration of Competing Interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author's Contribution

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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